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The Theory of Determinants in the Historical Order of its Development. By Thomas Muir, M.A., LL.D.

PART I. *Determinants in General* (1812–1827).

(Continued from p. 518 of vol. xiv.)

Up to this point a thorough understanding of the notation

$$\left(a^{(p)}_{1.P}\right)$$

is the one essential. Taking the particular instance

$$\left(a^{(2)}_{1.10}\right)$$

we first call to mind that it is an abbreviation for the determinant whose first row has for its last "terme" the determinant

$$a^{(2)}_{1.10},$$

—that is to say, an abbreviation for the system whose determinant we should nowadays write in the form

$$\begin{vmatrix} a^{(2)}_{1.1} & a^{(2)}_{1.2} & \cdots & a^{(2)}_{1.10} \\ a^{(2)}_{2.1} & a^{(2)}_{2.2} & \cdots & a^{(2)}_{2.10} \\ \cdot & \cdot & \cdot & \cdot \\ a^{(2)}_{10.1} & a^{(2)}_{10.2} & \cdots & a^{(2)}_{10.10} \end{vmatrix}.$$

The next point is to realise what determinants are denoted by

$$a^{(2)}_{1.1},\quad a^{(2)}_{1.2},\ \ldots\ldots\ldots\ldots$$

Now the number 10 being of necessity a combinatorial, and, as the figure in brackets above it indicates, of the form

$$\frac{n(n-1)}{1.2},$$

we see that n must be 5, and that the said determinants are all derived from

$$\begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} & a_{1\cdot 3} & a_{1\cdot 4} & a_{1\cdot 5} \\ a_{2\cdot 1} & a_{2\cdot 2} & a_{2\cdot 3} & a_{2\cdot 4} & a_{2\cdot 5} \\ a_{3\cdot 1} & a_{3\cdot 2} & a_{3\cdot 3} & a_{3\cdot 4} & a_{3\cdot 5} \\ a_{4\cdot 1} & a_{4\cdot 2} & a_{4\cdot 3} & a_{4\cdot 4} & a_{4\cdot 5} \\ a_{5\cdot 1} & a_{5\cdot 2} & a_{5\cdot 3} & a_{5\cdot 4} & a_{5\cdot 5} \end{vmatrix} .$$

The details of the process of derivation are recalled in connection with the interpretation of the pairs of suffixes. A requisite preliminary is to form all the different pairs of the numbers 1, 2, 3, 4, 5; arrange them in the order

$$12,\ 13,\ 14,\ 15,\ 23,\ 24,\ 25,\ 34,\ 35,\ 45\,;$$

and then number them

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\,.$$

These last are the numbers from which the suffixes are taken, and what each one as a suffix refers to, is the combination under which it is here placed. For example, the first suffix in $a^{(2)}_{1.1}$ refers to the combination 1 2, and implies the deletion of all the rows of the above determinant of the fifth order except the 1st and 2nd; the second suffix refers to the same combination, and implies the deletion of all the columns except the 1st and 2nd; and the symbol as a whole thus comes to stand for

$$\begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} \\ a_{2\cdot 1} & a_{2\cdot 2} \end{vmatrix} . \qquad \text{(XLI. 2.)}$$

Interpreting $a^{(2)}_{1.2}$, $a^{(2)}_{1.3}$, ... in the same way, we see that

$$\left(a^{(2)}_{1.10}\right)$$

is a compact notation for the system of which the determinant is

$$\begin{vmatrix} |a_{1\cdot 1}a_{2\cdot 2}|, & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ , & |a_{1\cdot 4}a_{2\cdot 5}| \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ , & |a_{3\cdot 4}a_{4\cdot 5}| \\ |a_{3\cdot 1}a_{5\cdot 2}|, & \cdot\ \cdot\ \cdot\ , & |a_{3\cdot 3}a_{5\cdot 4}|, & |a_{3\cdot 3}a_{5\cdot 5}|, & |a_{3\cdot 4}a_{5\cdot 5}| \\ |a_{4\cdot 1}a_{5\cdot 2}|, & \cdot\ \cdot\ \cdot\ , & |a_{4\cdot 3}a_{5\cdot 4}|, & |a_{4\cdot 3}a_{5\cdot 5}|, & |a_{4\cdot 4}a_{5\cdot 5}| \end{vmatrix} .$$

Similarly

$$\left(a^{(3)}_{1.10}\right)$$

stands for the system of which the determinant is

$$\left|\begin{array}{lll}
|a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}|, & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & , \; |a_{1\cdot3}a_{2\cdot4}a_{3\cdot5}| \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & , \;\cdot\;\cdot\;\cdot \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & , \; |a_{2\cdot3}a_{3\cdot4}a_{5\cdot5}| \\
|a_{2\cdot1}a_{4\cdot2}a_{5\cdot3}|, & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \;,\; |a_{2\cdot2}a_{4\cdot4}a_{5\cdot5}| & , \; |a_{2\cdot3}a_{4\cdot4}a_{5\cdot5}| \\
|a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|, & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \;,\; |a_{3\cdot2}a_{4\cdot4}a_{5\cdot5}| & , \; |a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}|
\end{array}\right|.$$

and which is called the "complementary derived system." (XLI. 3.) To every "terme" of the latter there corresponds a "terme" of the former, the one "terme" consisting exactly of those a's of the original determinant which are awanting in the other. This relationship Cauchy goes on to mark by means of a name and a notation. He calls two such "termes," $|a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|$ and $|a_{1\cdot4}a_{2\cdot5}|$ for example, "termes complémentaires des deux systèmes;" (XLI. 4) and if the symbol for the one be by previous agreement

$$a^{(p)}_{\mu.\pi}$$

the symbol for the other is made *

$$a^{(n-p)}_{\mathrm{P}-\mu+1\,.\,\mathrm{P}-\pi+1} \qquad \text{(XLI. 5.)}$$

As for the signs of the "termes" in "derived systems," Cauchy's words are (p. 98)—

> "En général, il est facile de voir que le produit de deux termes complémentaires pris à volonté est toujours, au signe près, une portion de ce même déterminant (D_n). Cela posé, étant donné le signe de l'un de ces deux termes, on déterminera celui de l'autre par la condition que leur produit soit affecté du même signe que la portion correspondante du déterminant D_n."

All these preliminaries having been settled, the weighty matters of the section are entered on. The first of these is a complete and

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* If Cauchy had adopted a slightly different principle for determining the order of combinations, the μ^{th} combination of p things and the $(\mathrm{P}-\mu+1)^{\text{th}}$ combination of $n-p$ things would have been mutually exclusive, and the convention here made in regard to notation would have been unnecessary.

perfectly accurate statement of the expansion-theorem, known by the name of Laplace, but which, as we have seen, Laplace and even Bézout who followed him were very far from fully formulating. The passage is of the greatest interest. No better example could be chosen to illustrate the powerful grasp which Cauchy had of the subject. What Laplace and Bézout laboured at, lengthily expounding one special case after another, Cauchy sets forth with ease and in all its generality in the space of a page. His words are (p. 99)—

> " On a fait voir dans le § 3^e^ que la fonction symétrique alternée
>
> $$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}\ldots a_{n\cdot n}) = D_n$$
>
> était équivalente à celle-ci
>
> $$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n-1\cdot}\quad\;).\, a_{n\cdot n}].$$
>
> On fera voir de même qu'elle est encore équivalente à
>
> $$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}).\, S(\pm a_{p+1\cdot p+1}\ldots a_{n-1\cdot n-1}a_{n\cdot n})],$$
>
> les opérations indiquées par le signe S pouvant être considérées comme relatives, soit aux premiers, soit aux seconds indices. On a d'ailleurs par ce qui précède
>
> $$S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) = \pm a^{(p)}_{1.1},$$
>
> $$S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n}) = \pm a^{(n-p)}_{P.P}.$$
>
> Enfin les signes des quantités de la forme
>
> $$a^{(p)}_{1.1}\;,\quad a^{(n-p)}_{P.P}$$
>
> doivent être tels que les produits semblables à
>
> $$a^{(p)}_{1.1}a^{(n-p)}_{P.P}$$
>
> soient dans le déterminant D_n affectés du signe +. Cela posé, il résulte de l'équation
>
> $$D_n = S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}).\, S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n})],$$
>
> que D_n est la somme de plusieurs produits de la forme
>
> $$a^{(p)}_{1.1}\; a^{(n-p)}_{P.P}.$$
>
> Selon que pour obtenir ces différens produits on échangera

> entre eux les premiers ou les seconds indices du système $(a_{1\cdot n})$, on trouvera ou l'équation
>
> $$\mathrm{D}_n = a^{p}_{1.1}\, a^{(n-p)}_{\mathrm{P.P}} + a^{(p)}_{2.1}\, a^{(n-p)}_{\mathrm{P-1.P}} + \qquad + a^{(p)}_{\mathrm{P.1}}\, a^{(n-p)}_{\mathrm{1.P}},$$
>
> ou celle-ci
>
> $$\mathrm{D}_n = a^{p}_{1.1}\, a^{(n-p)}_{\mathrm{P.P}} + a^{(p)}_{1.2}\, a^{(n-p)}_{\mathrm{P.P-1}} + \dots + a^{(p)}_{\mathrm{1.P}}\, a^{(n-p)}_{\mathrm{P.1}}.$$
>
> On aura de même en général les deux équations
>
> $$\mathrm{D}_n = a^{(p)}_{1.\pi}\, a^{(n-p)}_{\mathrm{P.P}-\pi+1} + a^{(p)}_{2.\pi}\, a^{(n-p)}_{\mathrm{P-1.P}-\pi+1} + \dots + a^{(p)}_{\mathrm{P}.\pi}\, a^{(n-p)}_{\mathrm{1.P}-\pi+1},$$
>
> $$\mathrm{D}_n = a^{(p)}_{\mu.1}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}} + a^{(p)}_{\mu.2}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}} + \dots + a^{(p)}_{\mu.\mathrm{P}}\, a^{(n-p)}_{\mathrm{P}-\mu+1.1}.$$
>
> Ces deux équations sont comprises dans la suivante
>
> $$\mathrm{D}_n = \mathrm{S}^{\mathrm{P}}\Big(a^{(p)}_{\mu.\pi}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\Big), \qquad \text{(XIV. 4.)}$$
>
> qui a lieu également, soit que l'on considère le signe S comme relatif à l'indice μ, soit qu'on le considère comme relatif à l'indice π."

Taking as an illustration the case where $n=5$, $p=2$, and $\pi=7$ (that is, the ordinal number corresponding to the pair 2 5, of the suffixes 1, 2, 3, 4, 5), and translating literally from Cauchy's notation into our own, we have

$$|a_{11}a_{22}a_{33}a_{44}a_{55}| = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{54}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{54}| + \dots$$
$$\dots\dots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{34}|.$$

With the same certainty of touch and with still greater conciseness, all the identities directly obtainable by Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes,* are formulated as one general identity, and established on a proper basis. The paragraph is (p. 100)—

> "D_n étant une fonction symétrique alternée des indices du système $(a_{1\cdot n})$ doit se réduire à zéro, lorsqu'on y remplace un de ces indices par un autre. Si l'on opère de semblables remplacemens à l'égard des indices qui occupent la première place dans le système $(a_{1\cdot n})$, et qui entrent dans la combinaison (μ); cette même combinaison se trouvera transformée en une autre que je désignerai par (ν), et $a^{(p)}_{\mu.\pi}$ sera changé en $a^{(p)}_{\nu.\pi}$. D'ailleurs, en supposant le signe S relatif à π, on a

$$\mathrm{D}_n = \mathrm{S}^{\mathrm{P}}\Big(a^{(p)}_{\mu.\pi}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\Big)\,;$$

ou aura donc par suite

$$0 = \mathrm{S}^{\mathrm{P}}\Big(a^{(p)}_{\nu.\pi}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\Big). \quad (\text{XII. } 7;\ \text{XXIII. } 3).$$

On aurait de même, en supposant le signe S relatif à l'indice μ, et en désignant par (τ) une nouvelle combinaison différente de (π)

$$0 = \mathrm{S}^{\mathrm{P}}\Big(a^{(p)}_{\mu.\tau}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\Big)." \quad (\text{XII. } 7;\ \text{XXIII. } 3).$$

As this theorem is twin with the preceding, it is best to illustrate it by the same special case. By so doing, indeed, both theorems become more readily grasped and their details better understood. Taking then as before $n = 5$, $p = 2$ and $\pi = 7$, we first form the determinants which Cauchy would have denoted by

$$a^{(2)}_{1.7}\,,\ a^{(2)}_{2.7}\,,\ \ldots\ldots,\ a^{(2)}_{10.7}\,,$$

and which we denote by

$$|a_{12}a_{25}|\,,\ |a_{12}a_{35}|\,,\ \ldots\ldots,\ |a_{42}a_{55}|\,.$$

Next, for cofactors, we form the determinants which are complementary, not of these, as in the preceding theorem, but of the members of one of the nine other groups corresponding to the values 1, 2, 3, 4, 5, 6, 8, 9, 10 of π,—say the group

$$a^{(2)}_{1.6}\,,\ a^{(2)}_{2.6}\,,\ \ldots\ldots,\ a^{(2)}_{10.6}\,.$$

These complementaries being

$$|a_{31}a_{43}a_{55}|\,,\ |a_{21}a_{43}a_{55}|\,,\ \ldots\ldots,\ |a_{11}a_{23}a_{35}|\,,$$

we have the desired identity

$$0 = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{55}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{55}| \ldots\ldots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{35}|\,,$$

the right-hand side of which is nothing more than an expansion of the zero determinant which arises from the determinant $|a_{11}a_{22}a_{33}a_{44}a_{55}|$ "lorsqu'on y remplace un des indices par un autre," viz., the second 4 by 5.

With the help of these two theorems a third theorem of almost equal importance is derived, viz., regarding the product of the determinants of two complementary systems. Denoting the determinant of the system

$$\left(a_{1.\mathrm{P}}^{(p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(p)},$$

and that of the complementary system

$$\left(a_{1.\mathrm{P}}^{(n-p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(n-p)},$$

and multiplying the two determinants together, we see with Cauchy that by (XIV. 4) the principal "termes" of the resulting determinant are each equal to

$$\mathrm{D}_n,$$

and by (XII. 7) all the other "termes" are equal to zero. Consequently

$$\mathrm{D}_{\mathrm{P}}^{(p)} \,.\, \mathrm{D}_{\mathrm{P}}^{(n-p)} = (\mathrm{D}_n)^{\mathrm{P}} \qquad \text{(XLII.)}$$

As an example of this theorem, it may be added that the product of the two determinants printed above (p. 482–3) to illustrate the notation

$$\left(a_{1.\mathrm{P}}^{(p)}\right),$$

that is to say, the determinants of the systems

$$\left(a_{1.10}^{(2)}\right), \left(a_{1.10}^{(3)}\right),$$

is equal to

$$|a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}|^{10}.$$

In connection with all the three theorems, the special case, $p=1$, is given, so that their relation to previously well-known theorems (VI., XII., XXI.) may be noted. It is also pointed out, that when in the third theorem n is even and $p=\frac{1}{2}n$, the result takes the interesting form

$$\mathrm{D}_{\mathrm{P}}^{\left(\frac{n}{2}\right)} = (\mathrm{D}_n)^{\frac{\mathrm{P}}{2}}, \qquad \text{(XLII. 2).}$$

This brings us to the last section of the memoir, the fourth, bearing the heading

Des Systèmes d'Equations dérivées et de leur Déterminans.

What it is concerned with is the relations subsisting between a "derived system" of the product-determinant

$$\begin{vmatrix} m_{1\cdot1} & m_{1\cdot2} & \ldots & m_{1\cdot n} \\ m_{2\cdot1} & m_{2\cdot2} & \ldots & m_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ m_{n\cdot1} & m_{n\cdot2} & \ldots & m_{n\cdot n} \end{vmatrix},$$

and the corresponding "derived systems" of the factors

$$\begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} & \ldots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots & a_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \ldots & a_{n\cdot n} \end{vmatrix} , \quad \begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} & \ldots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots & a_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \ldots & a_{n\cdot n} \end{vmatrix};$$

in other words, the relations which must connect the systems

$$\left(a^{(p)}_{1.\mathrm{P}}\right),\ \left(a^{(p)}_{1.\mathrm{P}}\right),\ \left(m^{(p)}_{1.\mathrm{P}}\right)$$

by reason of the relations

$$\Sigma[\mathrm{S}^n(a_{\nu\cdot1}\,a_{\mu\cdot1}) = m_{\mu.\nu}]$$

(given in full above on p. 513, Vol. XIV.) which connect the systems

$$(a_{1\cdot n})\ ,\ (a_{1\cdot n})\ ,\ (m_{1\cdot n}).$$

First of all, attention is concentrated on a single "terme" of the system

$$\left(m^{(p)}_{1.\mathrm{P}}\right),$$

or, as we should nowadays say, on a minor of the product-determinant. The process of reasoning, which occupies about four quarto pages, is exactly analogous to that previously followed in dealing with the product-determinant itself; and the result obtained is

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\,a^{(p)}_{\mu.1}\right), \quad . \quad . \quad . \quad (\text{XVIII. } 5),$$

where S^{P} is meant to indicate that the terms on the right-hand side are got by changing the second suffixes into 2, 3, 4, . . . , P in succession. Speaking roughly and in modern phraseology, we may say that this means that

Any minor of a product-determinant is expressible as a sum of products of minors of the two factors. (XVIII. 5.)

Cauchy then proceeds (p. 107)—

"Si dans cette équation [XVIII. 5] on donne successivement à

μ et à ν toutes les valeurs entières depuis 1 jusqu'à P, on aura un système d'équations symétriques de l'ordre P, que l'on pourra représenter par le symbole

$$(63) \qquad \Sigma\left\{ S^{P}\left(a^{(p)}_{\nu.1}\; a^{(p)}_{\mu.1}\right) = m^{(p)}_{\mu.\nu} \right\},$$

P étant toujours égal à

$$\frac{n(n-1)\;.\;.\;.\;.\;.\;(n-p+1)}{1\;.\;2\;.\;3\;.\;.\;.\;p}\;.$$

Pour déduire des équations

$$\Sigma\left[S^{n}(a_{\nu\cdot 1}\; a_{\mu\cdot 1}) = m_{\mu\cdot\nu}\right]$$

les équations (63), il suffit évidemment de remplacer les trois systèmes de quantités

$$(a_{1\cdot n}) \quad , \quad (a_{1\cdot n}) \quad , \quad (m_{1\cdot n})$$

par les systèmes dérivées de même ordre

$$\left(a^{(p)}_{1\cdot P}\right) , \left(a^{(p)}_{1.P}\right) , \left(m^{(p)}_{1.P}\right).$$

Je dirai pour cette raison que le second système d'équations est dérivé du premier." (XLI. 6.)

The close outward resemblance here noted between the original and the derived system of connecting equations is due of course to the choice of the notation

$$a^{p)}_{1.P}$$

for the minors of the determinant

$$S \pm (a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n\cdot n}) ,$$

and is so far a recommendation of that notation.

From the system of equations (63) two deductions follow immediately. In regard to the first Cauchy's words are (p. 108)—

"Désignons par

$$\delta^{(p)}_{P} \quad , \quad D^{(p)}_{P} \quad , \quad M^{(p)}_{P}$$

les déterminans des trois systémes

$$\left(a^{(p)}_{1.P}\right) , \left(a^{(p)}_{1.P}\right) , \left(m^{(p)}_{1.P}\right) ;$$

on aura en vertu des équations (63)

$$(65) \qquad M^{(p)}_{P} = D^{(p)}_{P}\, \delta^{(p)}_{P} \,.\text{"} \qquad \text{(XLIII.)}$$

The enunciation of this in modern phraseology would be—

Any compound of a product-determinant is equal to the product of the corresponding compounds of the two factors. (XLIII.)

The next deduction is stated equally succinctly (p. 109)—

> "Si l'on ajoute entre elles les équations (63) on aura la suivante,
>
> $$(66) \qquad \mathrm{S}^{\mathrm{P}}\left\{\mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\mu.\nu}\right)\mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\mu.\nu}\right)\right\} = \mathrm{S}^{\mathrm{P}}\mathrm{S}^{\mathrm{P}}\left(m^{(p)}_{\mu.\nu}\right), \qquad (\text{XXX. } 2)$$
>
> le premier signe S, c'est-à-dire le signe extérieur, étant relatif à l'indice ν, et les autres, c'est-à-dire les signes intérieurs, étant relatifs à l'indice μ."

This (66) corresponds to (XL.) as (65) corresponds to the multiplication theorem

$$\mathrm{M}_n = \mathrm{D}_n\delta_n,$$

the transition from the general to the particular being effected in both cases by putting $p = 1$.

With these deductions, the 4th Section practically comes to an end; but one or two results, intentionally omitted in the account of the 2nd Section because they seemed to belong naturally to the 4th, fall now to be noted.

The first is very simple. It arises (p. 91) from observing that

$$\begin{aligned} (\mathrm{D}_n)^{n-1} \times (\delta_n)^{n-1} &= (\mathrm{D}_n\delta_n)^{n-1}, \\ \text{and } \therefore \qquad &= (\mathrm{M}_n)^{n-1} \end{aligned}$$

by the multiplication-theorem. The result (XXI. 2) above (p. 110), is then thrice applied, and a theorem at once takes shape, which in later times we find enunciated as follows:—

The adjugate of the product-determinant is equal to the product of the adjugates of the two factors. (XLIII. 2.)

It is not noted, however, by Cauchy that this is but a case of XLIII., viz., where $p = n - 1$.

The next is

$$\begin{aligned} &\Sigma\,[\mathrm{S}^n(m_{1\cdot\nu}\ b_{1\cdot\mu}) = \mathrm{D}_n a_{\nu\cdot\mu}], \\ \text{or} \qquad &\Sigma\,[\mathrm{S}^n(m_{\mu\cdot 1}\ \beta_{1\cdot\nu}) = \delta_n a_{\mu.\nu}] \qquad (\text{XLIV.}) \end{aligned}$$

It is nothing more than the result of solving the $n.n$ equations

$$(33) \quad \Sigma\,[\mathrm{S}^n(a_{\nu\cdot 1}\ a_{\mu\cdot 1}) = m_{\mu\cdot\nu}]$$

first, in columns, for all the α's, and secondly, in rows for all the a's.

The last is

$$\Sigma\,[\mathrm{S}^n(a_{1\cdot\mu}\;r_{\nu\cdot 1}) = \delta_n b_{\nu\cdot\mu}]$$

$$\text{or}\qquad \Sigma\,[\mathrm{S}^n(a_{1\cdot\nu}\;r_{1\cdot\mu}) = \mathrm{D}_n\beta_{\mu\cdot\nu}] \qquad (\textsc{xliv}.\ 2)$$

where $(r_{1\cdot n})$ is the system adjugate to $(m_{1\cdot n\cdot})$. It is obtained from the $n.n$ equations (XLIV.) just as they were obtained from the $n.n$ equations (33), use being made of the theorem

$$\mathrm{M}_n = \mathrm{D}_n\delta_n.$$

In concluding, Cauchy refers to Binet's researches on similar matters. Most of what he says in regard to them has already been given (see p. 498, Vol. XIV. above). The rest of it is as follows (p. 111):—

> "Il [Binet] me dit en outre qu'il avait généralisé le théorème dont il s'agit $[\mathrm{M}_n = \mathrm{D}_n\delta_n]$, en substituant au produit de deux résultantes des sommes de produits de même espèce. J'avais dès-lors déjà démontré le théorème suivant :
>
> *D'un système quelconque d'équations symétriques on peut déduire cinq autres systèmes du même ordre ; mais on n'en saurait déduire un plus grand nombre.*
>
> J'ai démontré depuis à l'aide des méthodes précédentes cet autre théorème :
>
> *D'un système quelconque d'équations symétriques de l'ordre* n, *on peut toujours déduire deux systèmes d'équations symétriques de l'ordre*
>
> $$\frac{n(n-1)}{2},$$
>
> *deux systèmes d'équations symétriques de l'ordre*
>
> $$\frac{n(n-1)(n-2)}{1.2.3}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad .$$
>
> En ajoutant entre elles les équations symétriques comprises dans un même système, on obtient, comme on l'a vu, les formules (50), (51) et (70) qui me paraissent devoir être semblables à celles dont M. Binet m'a parlé."

The last sentence here raises an important question for the historian to settle, viz., whether Cauchy is to share with Binet the credit of the generalisation of the multiplication-theorem. The identities on which the claim is based are—

$$S^n\{S^n(a_{\mu\cdot\nu})\quad S^n(a_{\mu\cdot\nu})\} = S^nS^n(m_{\mu\cdot}{}^{\nu}) \qquad (50)$$

$$S^n\{S^n(\beta_{\mu\cdot\nu})\quad S^n(b_{\mu\cdot\nu})\} = S^nS^n(r_{\mu\cdot\nu}) \qquad (51)$$

$$S^P\left\{S^P\left(a^{(p)}_{\mu.\nu}\right)S^P\left(a^{(p)}_{\mu.\nu}\right)\right\} = S^PS^P\left(m^{(p)}_{\mu.\nu}\right) \qquad (70)$$

The first of these, given formerly (p. 516) in the uncontracted form

$$\begin{aligned}
&(a_{1\cdot1}+a_{2\cdot1}+\ .\ .\ .\ +a_{n\cdot1})\,(a_{1\cdot1}+a_{2\cdot1}+\ .\ .\ .\ +a_{n\cdot1})\\
&+(a_{1\cdot2}+a_{2\cdot2}+\ .\ .\ .\ +a_{n\cdot2})\,(a_{1\cdot2}+a_{2\cdot2}+\ .\ .\ .\ +a_{n\cdot2})\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
&+(a_{1\cdot n}+a_{2\cdot n}+\ .\ .\ .\ +a_{n\cdot n})\,(a_{1\cdot n}+a_{2\cdot n}+\ .\ .\ .\ +a_{n\cdot n})\\
=\ &\ m_{1\cdot1}+m_{2\cdot1}+\ .\ .\ .\ +m_{n\cdot1}\\
&+m_{1\cdot2}+m_{2\cdot2}+\ .\ .\ .\ +m_{n\cdot2}\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\\
&+m_{1\cdot n}+m_{2\cdot n}+\ .\ .\ .\ +m_{n\cdot n}
\end{aligned}$$

where $m_{\mu.\nu}=a_{\mu\cdot1}a_{\nu\cdot1}+a_{\mu\cdot2}a_{\nu\cdot2}+\ .\ .\ .\ +a_{\mu\cdot n}a_{\nu\cdot n}$,

may be at once left out of consideration; it is not even a case of the multiplication-theorem. Cauchy, we may be sure, mentioned it only because it is the first of the series to which (51) and (70) belong. The next concerns the systems

$$(\beta_{1\cdot n})\ ,\ (b_{1\cdot n})\ ,\ (r_{1\cdot n})$$

adjugate to the systems

$$(a_{1\cdot n})\ ,\ (a_{1\cdot n})\ ,\ (m_{1\cdot n})$$

dealt with in (50). It indeed is comparable with Binet's theorem; but as it is only a case of (70),—the minors in (70) being of any order whatever, whereas in (51) they are the principal minors,—we may without loss pass it over. Directing our attention, then, to (70) let us for the sake of greater definiteness take the case where $n=5$ and $p=2$, and where consequently $P=\dfrac{5\ .\ 4}{1\ .\ 2}=10$. The theorem then becomes

$$S^{10}\left\{S^{10}\left(a^{(2)}_{\mu.\nu}\right)S^{10}\left(a^{(2)}_{\mu.\nu}\right)\right\} = S^{10}S^{10}\left(m^{(2)}_{\mu.\nu}\right).$$

For the purpose of comparison with Binet's result, it is absolutely necessary, however, to depart from this exceedingly condensed mode of statement. Remembering that the inner S's refer always to the first suffix, and the outer to the second suffix, we obtain the more developed form

$$\begin{aligned}
&\left(\alpha^{(2)}_{1.1}+\alpha^{(2)}_{2.1}+\ .\ .\ .\ +\alpha^{(2)}_{10.1}\right)\left(a^{(2)}_{1.1}+a^{(2)}_{2.1}+\ .\ .\ .\ +a^{(2)}_{10.1}\right)\\
+&\left(\alpha^{(2)}_{1.2}+\alpha^{(2)}_{2.2}+\ .\ .\ .\ +\alpha^{(2)}_{10.2}\right)\left(a^{(2)}_{1.2}+a^{(2)}_{2.2}+\ .\ .\ .\ +a^{(2)}_{10.2}\right)\\
+&\ .\\
+&\left(\alpha^{(2)}_{1.10}+\alpha^{(2)}_{2.10}+\ .\ .\ .\ +\alpha^{(2)}_{10.10}\right)\left(a^{(2)}_{1.10}+a^{(2)}_{2.10}+\ .\ .\ .\ +a^{(2)}_{10.10}\right)\\
=\ \ &m^{(2)}_{1.1}+m^{(2)}_{2.1}+\ .\ .\ .\ +m^{(2)}_{10.1}\\
+&m^{(2)}_{1.2}+m^{(2)}_{2.2}+\ .\ .\ .\ +m^{(2)}_{10.2}\\
+&\ .\ .\ .\ .\ .\ .\\
+&m^{(2)}_{1.10}+m^{(2)}_{2.10}+\ .\ .\ .\ +m^{(2)}_{10.10}\,.
\end{aligned}$$

Interpreting now the suffixes and superfixes of the α's, a's, and m's, after the manner already described,—any suffix r signifying along with the superfix (2) the r^{th} combination of two numbers taken from 1, 2, 3, 4, 5, we finally reach the suitable form

$$\begin{aligned}
&\{\,|\alpha_{1\cdot1}\alpha_{2\cdot2}|+|\alpha_{1\cdot1}\alpha_{3\cdot2}|+\ldots\ldots+|\alpha_{3\cdot1}\alpha_{4\cdot2}|+|\alpha_{3\cdot1}\alpha_{5\cdot2}|+|\alpha_{4\cdot1}\alpha_{5\cdot2}|\,\}\\
&\{\,|a_{1\cdot1}a_{2\cdot2}|+|a_{1\cdot1}a_{3\cdot2}|+\ldots\ldots+|a_{3\cdot1}a_{4\cdot2}|+|a_{3\cdot1}a_{5\cdot2}|+|a_{4\cdot1}a_{5\cdot2}|\,\}\\
+&\ .\\
+&\{\,|\alpha_{1\cdot3}\alpha_{2\cdot5}|+|\alpha_{1\cdot3}\alpha_{3\cdot5}|+\ldots\ldots+|\alpha_{3\cdot3}\alpha_{4\cdot5}|+|\alpha_{3\cdot3}\alpha_{5\cdot5}|+|\alpha_{4\cdot3}\alpha_{5\cdot5}|\,\}\\
&\{\,|a_{1\cdot3}a_{2\cdot5}|+|a_{1\cdot3}a_{3\cdot5}|+\ldots\ldots+|a_{3\cdot3}a_{4\cdot5}|+|a_{3\cdot3}a_{5\cdot5}|+|a_{4\cdot3}a_{5\cdot5}|\,\}\\
+&\{\,|\alpha_{1\cdot4}\alpha_{2\cdot5}|+|\alpha_{1\cdot4}\alpha_{3\cdot5}|+\ldots\ldots+|\alpha_{3\cdot4}\alpha_{4\cdot5}|+|\alpha_{3\cdot4}\alpha_{5\cdot5}|+|\alpha_{4\cdot4}\alpha_{5\cdot5}|\,\}\\
&\{\,|a_{1\cdot4}a_{2\cdot5}|+|a_{1\cdot4}a_{3\cdot5}|+\ldots\ldots+|a_{3\cdot4}a_{4\cdot5}|+|a_{3\cdot4}a_{5\cdot5}|+|a_{4\cdot4}a_{5\cdot5}|\,\}\\
=\ \ &|m_{1\cdot1}m_{2\cdot2}|+|m_{1\cdot1}m_{3\cdot2}|+\ldots\ldots+|m_{3\cdot1}m_{4\cdot2}|+|m_{3\cdot1}m_{5\cdot2}|+|m_{4\cdot1}m_{5\cdot2}|\\
+&\ .\\
+&|m_{1\cdot3}m_{2\cdot5}|+|m_{1\cdot3}m_{3\cdot5}|+\ldots\ldots+|m_{3\cdot3}m_{4\cdot5}|+|m_{3\cdot3}m_{5\cdot5}|+|m_{4\cdot3}m_{5\cdot5}|\\
+&|m_{1\cdot4}m_{2\cdot5}|+|m_{1\cdot4}m_{3\cdot5}|+\ldots\ldots+|m_{3\cdot4}m_{4\cdot5}|+|m_{3\cdot4}m_{5\cdot5}|+|m_{4\cdot4}m_{4\cdot5}|\,,
\end{aligned}$$

where $$m_{\mu\cdot\nu}=a_{\mu\cdot1}a_{\nu\cdot1}+a_{\mu\cdot2}a_{\nu\cdot2}+\ .\ .\ .\ +a_{\mu\cdot5}a_{\nu\cdot5}\,.$$

The series of suffixes for the α's, a's, and m's are seen to be the same, the series of pairs of first suffixes in every row and the series of pairs of second suffixes in every column being

12, 13, 14, 15, 23, 24, 25, 34, 35, 45 ;

that is to say, the combinations arranged in ascending order, of the numbers 1, 2, 3, 4, 5, taken two at a time. On the first side of the identity are 10 products, and as both factors of each product contain 10 terms, the result of the multiplification would be to produce 1000 terms of the form

$$|a_{rp}a_{sq}|\cdot|a_{mp}a_{nq}|\,,$$

the whole expansion in fact being

$$\sum_{\substack{q=2\\ p<q}}^{q=5}\ \sum_{\substack{s=2\\ r<s}}^{s=5}\ \sum_{\substack{n=2\\ m<n}}^{n=5} |a_{rp}a_{sq}|\cdot|a_{mp}a_{nq}|\,.$$

On the right hand side are 100 terms of the form

$$|m_{rp}m_{sq}|\,,$$

and if a proof of the identity were wanted, we should only have to show that each of the 100 terms of the latter kind gives rise to a particular 10 terms of the former kind. This, too, it is interesting to note, Cauchy himself could have done. For example, the last of the 100 terms,

$$|m_{44}m_{55}|$$

$$= \begin{vmatrix} a_{41}a_{41}+a_{42}a_{42}+\ldots+a_{45}a_{45} & a_{41}a_{51}+a_{42}a_{52}+\ldots+a_{45}a_{55} \\ a_{51}a_{41}+a_{52}a_{42}+\ldots+a_{55}a_{45} & a_{51}a_{51}+a_{52}a_{52}+\ldots+a_{55}a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \times \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix}\cdot\begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} + \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix}\cdot\begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} + \ldots\ldots + \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix}\cdot\begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix},$$

which is nothing more than Cauchy's formula (62)

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\,a^{(p)}_{\mu.1}\right),$$

when we put $\mu = 10 = \nu$, and $p = 2$. Instead of 1000 terms on the left-hand side and 100 on the right, we should clearly have for the general theorem P^3 terms on the left and P^2 terms on the right, P be it remembered being the combinatorial

$$\frac{n(n-1)(n-2)\,.\,.\,.\,(n-p+1)}{1\,.\,2\,.\,3\,.\,.\,.\,p}.$$

Leaving Cauchy, let us now return to Binet, and in order that the comparison between the two may be complete, let us formally enunciate in all its generality the latter's theorem also. Binet himself did not do this. After dealing with the case in which the determinants involved are of the 2nd order, he merely added (p. 289)—

"On aura encore pour les intégrales

$$\Sigma\{\mathrm{S}(x,y',z'')\,\mathrm{S}(\xi,v',\zeta'')\},\quad \Sigma\{\mathrm{S}(t,x',y'',z''')\,\mathrm{S}(\tau,\xi',v'',\zeta''')\},\ \&\text{c.}$$

des résultats semblables, savoir,

$$\Sigma\{\mathrm{S}(x,y',z'')\,\mathrm{S}(\xi,v',\zeta'')\}$$

$$=\mathrm{S}_1\left\{\begin{array}{l}\quad\Sigma x\xi\Sigma yv\Sigma z\zeta+\Sigma y\xi\Sigma zv\Sigma x\zeta+\Sigma z\xi\Sigma xv\Sigma y\zeta\\ -\Sigma x\xi\Sigma zv\Sigma y\zeta-\Sigma y\xi\Sigma xv\Sigma z\zeta-\Sigma z\xi\Sigma yv\Sigma x\zeta,\end{array}\right.$$

$$\Sigma\{\mathrm{S}(t,x',y'',z''')\,\mathrm{S}(\tau,\xi',v'',\zeta''')\}$$

$$=\mathrm{S}_1\{\Sigma t\tau\Sigma x\xi\Sigma yv\Sigma z\xi+\Sigma x\tau\Sigma y\xi\Sigma tv\Sigma z\zeta+\&\text{c.}\}$$

&c."

With the help of modern phraseology, the general theorem thus intended to be indicated can be made sufficiently clear. Binet in effect says:—

Take s rectangular arrays each with m elements in the row and n elements in the column, m being greater than n, viz.—

$$\begin{array}{llll}(a_1)_{11}(a_1)_{12}\ldots(a_1)_{1m} & (a_1)_{21}(a_1)_{22}\ldots(a_1)_{2m} & \cdot\ \cdot & (a_1)_{s1}(a_1)_{s2}\ldots(a_1)_{sm}\\ (a_2)_{11}(a_2)_{12}\ldots(a_2)_{1m} & (a_2)_{21}(a_2)_{22}\ldots(a_2)_{2m} & \cdot\ \cdot & (a_2)_{s1}(a_2)_{s2}\ldots(a_1)_{sm}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (a_n)_{11}(a_n)_{12}\ldots(a_n)_{1m}, & (a_n)_{21}(a_n)_{22}\ldots(a_n)_{2m}, & \cdot\ \cdot & (a_n)_{s1}(a_n)_{s2}\ldots(a_n)_{sm}\end{array}$$

and other s rectangular arrays of the same kind, viz.—

$$\begin{array}{llll}(b_1)_{11}(b_1)_{12}\ldots(b_1)_{1m} & (b_1)_{21}(b_1)_{22}\ldots(b_1)_{2m} & \cdot\ \cdot & (b_1)_{s1}(b_1)_{s2}\ldots(b_1)_{sm}\\ (b_2)_{11}(b_2)_{12}\ldots(b_2)_{1m} & (b_2)_{21}(b_2)_{22}\ldots(b_2)_{2m} & \cdot\ \cdot & (b_2)_{s1}(b_2)_{s2}\ldots(b_2)_{sm}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (b_n)_{11}(b_n)_{12}\ldots(b_n)_{1m} & (b_n)_{21}(b_n)_{22}\ldots(b_n)_{2m} & \cdot & (b_n)_{s1}(b_n)_{s2}\ldots(b_n)_{sm}\end{array}$$

From each array, by taking every set of n columns, form $C_{m,n}$ determinants, arranging them in any order, provided it be the same for all the arrays. Add together all the 1st determinants formed from the first s arrays, and multiply the sum by the corresponding sum for the second s arrays; obtain the like product involving all the 2nd determinants, the like product involving all the 3rd determinants, and so on. Then, the sum of these products is equal to the sum of the products obtained by multiplying each array of the first set by each array of the second set.

Or we may put it alternatively as a formal proposition, thus:—

If s *rectangular arrays be taken, each with* m *elements in the row and* n *elements in the column,* m *being greater than* n, viz.

$$X_1, X_2, \ldots\ldots, X_s$$

and other s *rectangular arrays of the same kind,* viz.,

$$\Xi_1, \Xi_2, \ldots\ldots, \Xi_s;$$

and if the minor determinants of the n*th* *order formed from* X_1, X_2, $\ldots$, Ξ_{s-1}, Ξ_s *be*

$$\begin{array}{cccc} x_{11} & x_{12} & \ldots & x_{1C} \\ x_{21} & x_{22} & \ldots & x_{2,C} \\ \cdot & \cdot & \cdot & \cdot \\ x_{s1} & x_{s2} & \ldots & x_{s,C} \end{array} \qquad \begin{array}{cccc} \xi_{11} & \xi_{12} & \ldots & \xi_{1,C} \\ \xi_{21} & \xi_{22} & \ldots & \xi_{2,C} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{s1} & \xi_{s} & \ldots & \xi_{s,C} \end{array}$$

then

$$\begin{aligned} & (x_{11}+x_{21}+\ldots+x_{s1})(\xi_{11}+\xi_{21}+\ldots+\xi_{s1}) \\ & +(x_{12}+x_{22}+\ldots+x_{s2})(\xi_{12}+\xi_{22}+\ldots+\xi_{s2}) \\ & +\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & +(x_{1,C}+x_{2C}+\ldots+x_{s,C})(\xi_{1,C}+\xi_{2C}\ldots+\xi_{s,C}) \\ = {} & (X_1+X_2+\ldots+X_s)(\Xi_1+\Xi_2+\ldots+\Xi_s) \end{aligned}$$

where C stands for $C_{m,n}$ *i.e.*, $m(m-1\ldots\ldots(m-n+1)/1.2.3\ldots\ldots n$.

Now, counting the terms here as we did in the case of Cauchy's theorem, we see that on the left-hand side there are C multiplications to be performed, each giving rise to $s \times s$ terms, and that therefore the full number of terms in the development of this side is

$$s^2C;$$

also that on the right-hand side the number is

$$s^2.$$

In Cauchy's theorem the corresponding numbers were found to be P^3 and P^2, P being not any whole number as s is, but like C a combinatorial. Without further investigation, we might consequently assert that, supposing the two theorems to be alike in other respects, Binet's must be the more general, the passage from it to Cauchy's being effected by taking $s = C$. A closer examination, however, will show that this is not the full measure of the difference between the two theorems as to generality. Not only must we specialise by putting $s = C$, but s must become C in a very special way. In order to make this clear, let us take the particular case of Binet's theorem which approximates as nearly as possible to the particular case of Cauchy's given above. In the latter the determinants were of the 2nd order; therefore to get the comparable case of Binet's theorem we must put $n = 2$. Again, since P in the particular case of Cauchy's theorem was 10, we must for the same purpose put

$$s = 10$$

$$\text{and } C_{m.n} = 10\,, \text{ and } \therefore m = 5\,.$$

The result is

$$\left\{ \begin{vmatrix} a_{11} a_{12} \\ b_{11} b_{12} \end{vmatrix} + \begin{vmatrix} a_{21} a_{22} \\ b_{21} b_{22} \end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1} a_{10,2} \\ b_{10,1} b_{10,2} \end{vmatrix} \right\} \left\{ \begin{vmatrix} \alpha_{11} \alpha_{12} \\ \beta_{11} \beta_{12} \end{vmatrix} + \ldots . + \begin{vmatrix} \alpha_{10,1} \alpha_{10,2} \\ \beta_{10,1} \beta_{10,2} \end{vmatrix} \right\}$$

$$+ \left\{ \begin{vmatrix} a_{11} a_{13} \\ b_{11} b_{13} \end{vmatrix} + \begin{vmatrix} a_{21} a_{23} \\ b_{21} b_{23} \end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1} a_{10,3} \\ b_{10,1} b_{10,3} \end{vmatrix} \right\} \left\{ \begin{vmatrix} \alpha_{11} \alpha_{13} \\ \beta_{11} \beta_{13} \end{vmatrix} + \ldots . + \begin{vmatrix} \alpha_{10,1} \alpha_{10,3} \\ \beta_{10,1} \beta_{10,3} \end{vmatrix} \right\}$$

$$+ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+ \left\{ \begin{vmatrix} a_{14} a_{15} \\ b_{14} b_{15} \end{vmatrix} + \begin{vmatrix} a_{24} a_{25} \\ b_{24} b_{25} \end{vmatrix} + \ldots . + \begin{vmatrix} a_{10,4} a_{10,5} \\ b_{10,4} b_{10,5} \end{vmatrix} \right\} \left\{ \begin{vmatrix} \alpha_{14} \alpha_{15} \\ \beta_{14} \beta_{15} \end{vmatrix} + \ldots . + \begin{vmatrix} \alpha_{10,4} \alpha_{10,5} \\ \beta_{10,4} \beta_{10,5} \end{vmatrix} \right\}$$

$$= \left\{ \begin{vmatrix} a_{11}\, a_{12} \ldots a_{15} \\ b_{11}\, b_{12} \ldots b_{15} \end{vmatrix} + \begin{vmatrix} a_{21}\, a_{22} \ldots a_{25} \\ b_{21}\, b_{22} \ldots b_{25} \end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1}\, a_{10,2} \ldots a_{10,5} \\ b_{10,1}\, b_{10,2} \ldots b_{10,5} \end{vmatrix} \right\} \times$$

$$\left\{ \begin{vmatrix} \alpha_{11}\, \alpha_{12} \ldots \alpha_{15} \\ \beta_{11} \beta_{12} \ldots \beta_{15} \end{vmatrix} + \begin{vmatrix} \alpha_{21}\, \alpha_{22} \ldots \alpha_{25} \\ \beta_{21} \beta_{22} \ldots \beta_{25} \end{vmatrix} + \ldots + \begin{vmatrix} \alpha_{10,1}\, \alpha_{10,2} \ldots \alpha_{10,5} \\ \beta_{10,1} \beta_{10,2} \ldots \beta_{10,5} \end{vmatrix} \right\},$$

the elements involved being 200 in number, and disposable in two sets of arrays—

$$\begin{matrix} a_{11}\ a_{12}\ \ldots\ a_{15} & a_{21}\ a_{22}\ \ldots\ a_{25} & & a_{10,1}\ a_{10,2}\ \ldots\ a_{10,5} \\ b_{11}\ b_{12}\ \ldots\ b_{15}\,, & b_{21}\ b_{22}\ \ldots\ b_{25}\,, & \ldots . & b_{10,1}\ b_{10,2}\ \ldots\ b_{10,5}\,, \end{matrix}$$

and

$$\begin{array}{ccc} \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} & \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25} & \cdots\cdots & \alpha_{10,1}\ \alpha_{10,2}\ \cdots\ \alpha_{10,5} \\ \beta_{11}\ \beta_{12}\ \cdots\ \beta_{15}, & \beta_{21}\ \beta_{22}\ \cdots\ \beta_{25}, & \cdots\cdots & \beta_{10,1}\ \beta_{10,2}\ \cdots\ \beta_{10,5}. \end{array}$$

In the corresponding identity of Cauchy, there are only 50 different elements, viz., the elements of the two square arrays—

$$\begin{array}{cc} a_{11}\ a_{12}\ \cdots\ a_{15} & \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} \\ a_{21}\ a_{22}\ \cdots\ a_{25} & \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{51}\ a_{52}\ \cdots\ a_{55}, & \alpha_{51}\ \alpha_{52}\ \cdots\ \alpha_{55}. \end{array}$$

Indeed,—and it is this which brings the comparison to a point,—if from the first of these square arrays we form 10 rectangular arrays by taking every possible pair of rows, thus using each row 4 times over, viz.,

$$\begin{array}{cccc} a_{11}\ a_{12}\ \cdots\ a_{15} & a_{11}\ a_{12}\ \cdots\ a_{15} & & a_{41}\ a_{42}\ \cdots\ a_{45} \\ & & \cdots\cdots & \\ a_{21}\ a_{22}\ \cdots\ a_{25}, & a_{31}\ a_{32}\ \cdots\ a_{35}, & , & a_{51}\ a_{52}\ \cdots\ a_{55}, \end{array}$$

and similarly from the α's form a second set of 10 arrays, viz.,

$$\begin{array}{cccc} \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15}, & \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} & & \alpha_{41}\ \alpha_{42}\ \cdots\ \alpha_{45} \\ & & \cdots\cdots & \\ \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25}, & \alpha_{31}\ \alpha_{32}\ \cdots\ \alpha_{35}, & , & \alpha_{51}\ \alpha_{52}\ \cdots\ \alpha_{55}; \end{array}$$

and then to these two special sets of arrays apply Binet's theorem, we obtain Cauchy's theorem. Regarding the two theorems in all their generality, the decision we have reached may therefore be expressed by saying that Binet's is a theorem concerning $2smn$ quantities, where s, m, n are any positive integers, and Cauchy's is a case of it in which $s=m(m-1)\ \ldots\ (m-n+1)/1\,.\,2\,.\,3\ \ldots\ n$, and in which, further, the number of different quantities involved is not

$$2\,.\,\frac{m(m-1)\ \ldots\ (m-n+1)}{1\,.\,2\ \ \ldots\ \ n}\times mn,$$

but by reason of repetitions is only

$$2m^2.$$

Although this decision is against Cauchy's claim as put by himself, it deserves to be noticed that, apparently by oversight, he failed to make his case as strong as he might have done. It will be remembered that Binet made two advances in the generalisation of the multiplication-theorem. In the first place, he gave the generalisation from which the multiplication-theorem is got by putting $m=n$, or, as we nowadays say, by substituting two square matrices for two rectangular matrices, and then he gave the theorem which we have been comparing with Cauchy's and which degenerates into his own first theorem when s is put equal to 1. Now the first of these generalisations Cauchy could justly have laid claim to. His identity (XVIII. 5) is not indeed stated or viewed as a generalisation of the multiplication-theorem, but it is unquestionably so in reality. Ostensibly the identity concerns any minor of a product-determinant, but every such minor is obtained by multiplying together two rectangular matrices, and, conversely, every determinant which is the product of two rectangular matrices may be viewed as a minor of the product of two determinants.

On looking back, however, at Cauchy's memoir as a whole, one cannot but be struck with admiration both at the quality and the quantity of its contents. Supposing that none of its theorems had been new, and that it had not even presented a single old theorem in a fresh light, the memoir would have been most valuable, furnishing as it did, to the mathematicians of the time, an almost exhaustive treatise on the theory of general determinants. It is not too much to say, although it may come to many as a surprise, that the ordinary text-books of determinants supplied to university students of the present day do not contain more of the general theory than is to be found in Cauchy's memoir of about eighty years ago. One apparently trivial instrument, which Cauchy had not received from his predecessors, and which he did not make for himself, viz., a notation for determinants whose elements had special values, is at the foundation of the whole difference between his treatise and those at present employed. When this want came to be supplied later on, the functions crept steadily into everyday use, and a fresh impetus was consequently given to the study of them. But if from the work of the said eighty years all researches regarding special forms of determinants be left out, and all investigations which ended in mere rediscoveries or in rehabilitations of old ideas, there is a surprisingly small proportion left. If one bears this in mind, and recalls the fact, temporarily set aside, that Cauchy, instead of being a compiler, presented the entire subject from a perfectly new point of view, added many results previously unthought of, and opened up a whole avenue of fresh investigation, one cannot but assign to him the place of highest honour among all the workers from 1693 to 1812. It is, no doubt, impossible to call him, as some have done, the formal founder of the theory. This honour is certainly due to Vandermonde, who, however, erected on the foundation comparatively little of a superstructure. Those who followed Vandermonde contributed, knowingly or unknowingly, only a stone or two, larger or smaller, to the building. Cauchy relaid the foundation, rebuilt the whole, and initiated new enlargements; the result being an edifice which the architects of to-day may still admire and find worthy of study.

RETROSPECT OF THE PERIOD 1693–1812.

From what has just been said by way of estimate of Cauchy's memoir, it will readily appear that a suitable opportunity has now presented itself for taking a general retrospect of the work done from the date at which the history commences. The system which has been pursued, of numbering the new advances made by each writer, enables us to do this very conveniently, and with a tolerable approximation to accuracy by means of a tabular form. The table, herewith annexed, so far explains itself. The authors' names, it will be seen, are arranged both vertically and horizontally in chronological order; and vertical and horizontal lines of separation are drawn so as to apportion to each name a gnomon-shaped space. The crediting of any entirely new result to an author is done by giving its number in Roman figures after his name in the vertical list. On the other hand, any mere modification, fresh presentment, or extension of a previously known result, is notified to the right of the original number of the result, and under the new writer's name in the horizontal series. Instead of the Arabic figures placed in the gnomon-shaped spaces, a cross or other uniform mark would have sufficed, but in order to increase the usefulness of the table, a number has been inserted, telling the page (counting from the commencement of the History) at which the result is to be found. For example, if we look to the space allotted to Bézout (1779), we find him credited with one entirely new result, numbered XXIII., and with some contribution to each of five previously known results, whose numbers are II., III., IV., XII., XIV.; and we likewise see that information regarding them all will be got at p. 53 of the History. Speaking generally, more importance ought to be attached to the existence of numbers at the corner of a gnomon than elsewhere, because these indicate fresh departures in the theory. Sometimes, however, a fresh departure may have been very trivial, the real advance being indicated by a number well removed from the corner of a subsequent gnomon. Thus if we examine the history of the multiplication-theorem (Nos. XVII., XVIII.), we find the first step in the direction of it credited by the table to Lagrange, and subsequent steps to Gauss, Binet, and Cauchy; whereas careful investigation at the pages mentioned shows that what Lagrange accomplished was of exceedingly little moment, in comparison with the magnificent generalisation of Binet and Cauchy. Again, it must be borne in mind that all the results numbered in Roman figures are not of equal importance, it being well known that one theorem in any mathematical subject may have vastly more influence on the after development of the subject than half a dozen others. Such imperfections, however, being allowed for, the table will be found to afford a very ready means of estimating with considerable accuracy the proportionate importance to be assigned to the various early investigators of the theory.

If we look for a moment, in conclusion, at the nationality of the authors, one outstanding fact immediately arrests attention, viz., that almost every important advance is due to the mathematicians of France. Were the contributions of Bézout, Vandermonde, Laplace, Lagrange, Monge, Binet, and Cauchy left out, there would be exceedingly little left to any one else, and even that little would be of minor interest.

GERGONNE (1813).

[Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré. *Annales de Mathématiques*, iv. pp. 148–155.]

This is such an exposition of the primary elements of the theory of determinants and their application to the solution of a set of simultaneous linear equations as might be given in the course of an hour's lecture. It is confessedly founded on Laplace's memoir of 1772; but, though the matter of it is thus not original, it is nevertheless noteworthy on account of its brevity, clearness, and elegance.

The word "inversion" is introduced to denote (III. 20) what Cramer called a "dérangement," and then by easy steps the reader is led up to the theorem regarding the interchange of two non-contiguous letters.

> "(9) Donc, si l'on permute entre elles deux lettres non consécutives, on changera nécessairement l'espèce du nombre des inversions. Soit en effet n le nombre des lettres intermédiaires à ces deux-la; on pourra d'abord porter la lettre la plus à gauche immédiatement à gauche de l'autre, ce qui lui fera parcourir n places; puis remettre cette dernière à la place de la première; et, comme elle sera obligée de passer par-dessus celle-ci, elle se trouvera avoir parcouru $n+1$ places. Le nombre total des places parcourues par les deux lettres sera donc $2n+1$, et conséquemment l'espèce du nombre des inversions se trouvera changée." (III. 21)

This, it must be noted, is not identical with Rothe's proposition on the same subject, Gergonne's n being different from Rothe's d.

The proof, that a determinant vanishes if two of the letters bearing suffixes be the same, proceeds on the same lines as Rothe's, but is put very shortly and not less convincingly as follows:—

> "Supposons, en effet, que l'on change h en g, sans toucher à g ni aux indices. Soient, pour un terme pris au hazard dans le polynôme, p et q les indices respectifs de g et h; ce polynôme, renfermant toutes les permutations, doit avoir un autre terme ne différant uniquement de celui-la qu'en ce que c'est h qui y porte l'indice p et g l'indice q; et de plus (9) ceux deux termes doivent être affectés de signes contraires; ils se détruiront donc, lorsqu'on changera h en g; et il en sera de même de tous les autres termes pris deux à deux." (XII. 8).

On putting "le polynôme D," *i.e.* the determinant $|a_1b_2c_3,\ldots|$ in the form

$$A_1a_1 + A_2a_2 + A_3a_3 + \ldots\ldots + A_ma_m,$$

this theorem of course leads at once to the identities

$$\left.\begin{array}{l} A_1 b_1 + A_2 b_2 + A_3 b_3 + \ldots + A_m b_m = 0 \\ A_1 c_1 + A_2 c_2 + A_3 c_3 + \ldots + A_m c_m = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

and these to the solution of m linear equations in m unknowns.

WRONSKI (1815).

[Philosophie de la Technie Algorithmique. Première Section, contenant la loi suprême et universelle des Mathématiques. Par Hoëné Wronski, pp. 175–181, &c. Paris.]

Here as in the *Réfutation* of 1812 "combinatory sums" make their appearance, as being necessary for the expression of the "loi suprême." Wronski's point of view is unaltered toward them. He now, however, calls them

Schin functions, (xv. 5)

from the letter formerly introduced to denote them, et "pour ne pas introduire de noms nouveaux"! Two or three pages are occupied with the statement of the recurrent law of formation (Bézout, 1764).

DESNANOT, P. (1819).

[Complément de la Théorie des Équations du Premier Degré, contenant . . . . . . Par P. Desnanot, Censeur au Collége Royal de Nancy, ... Paris.]

As far as can be gathered, Desnanot was acquainted with the writings of very few of his predecessors in the investigation of determinants. The only one he himself mentions is Bézout, and the first part of his work is in direct continuation of a topic which the latter had begun. His book is a marvel of laboured detail. No expositor could take more pains with his reader, space being held of no moment if clearness had to be secured. As might be expected, therefore, all that is really worth preserving of his work is but a small fraction of the 264 pages which he occupies in exposition.

The first chapter bears the heading

Recherche des Relations qui ont lieu entre le dénominateur et les numérateurs des valeurs générales des inconnues dans chaque système d'équations du premier degré ;

and, after a reference to the impossibility of obtaining any result in the case of one equation with one unknown, proceeds as follows :—

Si l'on a les deux équations

$$ax + by = c\,, \quad a'x + b'y = c'\,,$$

elles donnent

$$x = \frac{cb' - bc'}{ab' - ba'}\,, \quad y = \frac{ac' - ca'}{ab' - ba'}\,;$$

nommant D le dénominateur commun, N et N′ les numérateurs des valeurs de x et de y, nous aurons

$$\mathrm{D} = ab' - ba'\,, \quad \mathrm{N} = cb' - bc'\,, \quad \mathrm{N}' = ac' - ca'\,.$$

Multiplions N par a, N′ par b et ajoutons, nous trouverons

$$a\mathrm{N} + b\mathrm{N}' = c(ab' - ba') = c\mathrm{D}\,;$$

donc

$$a\mathrm{N} + b\mathrm{N}' = c\mathrm{D}$$

Nous aurions de même, en multipliant N par a' et N′ par b', cette autre équation

$$a'\mathrm{N} + b'\mathrm{N}' = c'\mathrm{D}\,.\text{"}$$

With this may be compared Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par ellesmêmes* (see p. 456, Vol. XIV.).

Exactly the same method is followed with the set of equations

$$\left.\begin{array}{l} ax + by + cz = d \\ a'x + b'y + c'z = d' \\ a''x + b''y + c''z = d'' \end{array}\right\}.$$

Here fifteen relations are obtained, only seven of which are viewed as necessary, viz.,

$$\left.\begin{array}{l} (ab' - ba')\mathrm{N}' + (ac' - ca')\mathrm{N}'' = (ad' - da')\mathrm{D} \\ (ab'' - ba'')\mathrm{N}' + (ac'' - ca'')\mathrm{N}'' = (ad'' - da'')\mathrm{D} \end{array}\right\},$$

$$\left.\begin{array}{l} (da' - ad')\mathrm{N} + (db' - bd')\mathrm{N}' + (dc' - cd')\mathrm{N}'' = 0 \\ (da'' - ad'')\mathrm{N} + (db'' - bd'')\mathrm{N}' + (dc'' - cd'')\mathrm{N}'' = 0 \end{array}\right\},$$

$$\left.\begin{array}{l} a\,\mathrm{N} + b\,\mathrm{N}' + c\,\mathrm{N}'' = d\,\mathrm{D} \\ a'\,\mathrm{N} + b'\,\mathrm{N}' + c'\,\mathrm{N}'' = d'\,\mathrm{D} \\ a''\mathrm{N} + b''\mathrm{N}' + c''\mathrm{N}'' = d''\mathrm{D} \end{array}\right\}.$$

From a modern point of view there are but *two* which are really different, viz.,

$$|ab'|.|ac'd''| - |ac'|.|ab'd''| + |ad'|.|ab'c''| = 0$$

and

$$a|bc'd''| - b|ac'd''| + c|ab'd''| - d|ab'c''| = 0\,,$$

the twelve quantities concerned being

$$\begin{array}{cccc} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d''\,. \end{array}$$

The former is obtainable from Bézout's identity

$$|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0$$

by putting

$$f\,,f'\,,f'' = 0\,,0\,,1$$

and

$$e\,,e',\;e'' = a\,,a',a''.$$

The other, as is well known, comes from Vandermonde.

Before proceeding to the case of four unknowns, a notation is introduced in the following words (p. 6) :—

> "Soient a, b, c, d, f, g, h, etc. des lettres représentant des quantités quelconques ; k, l, m, p, q, r, etc. des indices d'accens qui doivent être placés à la droite des lettres. Au lieu de mettre ces indices comme des exposans, plaçons-les au-dessus des lettres qu'ils doivent affecter, de manière que $\overset{k}{a}$ désigne a affecté du nombre k d'accens ; que $\overset{k}{a}\overset{l}{b}$ indique le produit de $\overset{k}{a}$ par $\overset{l}{b}$; ainsi de suite. Représentons la quantité $\overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}$ par $\begin{pmatrix} k\ l \\ a\ b \end{pmatrix}$ de sorte que nous ayons cette équation
>
> $$\begin{pmatrix} k\ l \\ a\ b \end{pmatrix} = \overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}\,."$$

This being settled, the similar quantities of higher orders are defined by the equations

$$\begin{pmatrix} k\ l\ m \\ a\ b\ c \end{pmatrix} = \overset{m}{c}\begin{pmatrix} k\ l \\ a\ b \end{pmatrix} - \overset{l}{c}\begin{pmatrix} k\ m \\ a\ b \end{pmatrix} + \overset{k}{c}\begin{pmatrix} l\ m \\ a\ b \end{pmatrix},$$

$$\begin{pmatrix} k\ l\ m\ p \\ a\ b\ c\ d \end{pmatrix} = \overset{p}{d}\begin{pmatrix} k\ l\ m \\ a\ b\ c \end{pmatrix} - \overset{m}{d}\begin{pmatrix} k\ l\ p \\ a\ b\ c \end{pmatrix} + \overset{l}{d}\begin{pmatrix} k\ m\ p \\ a\ b\ c \end{pmatrix} - \overset{k}{d}\begin{pmatrix} l\ m\ p \\ a\ b\ c \end{pmatrix},$$

&c. &c. &c.

It is thus seen that Desnanot's definition is exactly the same as

Vandermonde's, and his notation essentially the same as Laplace's. To this definition and the proof of the theorem regarding the effect of the interchange of two indices or two letters seven pages are devoted, and then a fresh step is taken. The exact words of the original (pp. 13, 14) must be given, as they distinctly foreshadow a great theorem of later times.

> "14. Si nous développons cette expression
>
> $$\binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b}$$
>
> le résultat sera
>
> $$\binom{k\,m}{a\,b}\binom{l\,p}{a\,b};$$
>
> donc nous avons cette équation
>
> $$(A)\quad \binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b} = \binom{k\,m}{a\,b}\binom{l\,p}{a\,b}.$$
>
> 15. De cette formule je vais en déduire d'autres. Je dis que si j'introduis la lettre c dans les seconds facteurs de chaque terme et en même temps l'indice k, l'équation subsistera encore, et que j'aurai
>
> $$(B)\quad \binom{k\,l}{a\,b}\binom{k\,m\,p}{a\,b\,c} - \binom{k\,p}{a\,b}\binom{k\,m\,l}{a\,b\,c} = \binom{k\,m}{a\,b}\binom{k\,l\,p}{a\,b\,c}.$$
>
> L'égalité serait prouvée si en développant les deux membres, les quantités multipliées par la même lettre c, affectée d'indices égaux, étaient égales dans chaque membre; or j'ai
>
> $$\left.\begin{array}{l} {}^{p}_{c}\binom{k\,l}{a\,b}\binom{k\,m}{a\,b} \\ -{}^{m}_{c}\left(\binom{k\,l}{a\,b}\binom{k\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{k\,l}{a\,b}\right) \\ +{}^{k}_{c}\left(\binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b}\right) \\ -{}^{l}_{c}\binom{k\,p}{a\,b}\binom{k\,m}{a\,b} \end{array}\right\} = \left\{\begin{array}{l} +{}^{p}_{c}\binom{k\,m}{a\,b}\binom{k\,l}{a\,b} \\ \\ +{}^{k}_{c}\binom{k\,m}{a\,b}\binom{l\,p}{a\,b} \\ -{}^{l}_{c}\binom{k\,m}{a\,b}\binom{k\,p}{a\,b}. \end{array}\right.$$
>
> Les quantités multipliées par ${}^{p}_{c}$, ${}^{m}_{c}$ et ${}^{l}_{c}$ dans chaque membre sont égales entr'elles, c'est évident; et la formule (A) rend les coefficiens de ${}^{k}_{c}$ égaux; donc puisque dans (B), il n'y a que des termes multipliés par ${}^{p}_{c}$, ${}^{m}_{c}$, ${}^{k}_{c}$, et ${}^{l}_{c}$, je conclus que l'équation (B) est exacte."

Having thus shown that if in each of the second factors of the identity

$$|a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3| = 0 \qquad \text{(A)},$$

a new letter c be added and the index 1 be prefixed, the sign of equality may still be retained, so that we have a new identity

$$|a_1b_2||a_1b_3c_4| - |a_1b_3||a_1b_2c_4| + |a_1b_4||a_1b_2c_3| = 0 \qquad \text{(B)};$$

he then goes on to prove in the same fashion that the first factors of this derived identity may be treated in a similar way with impunity, viz., that they may be extended by the appending of the letter c with a new index 5, so that we have a further derived identity

$$|a_1b_2c_5||a_1b_3c_4| - |a_1b_3c_5||a_1b_2c_4| + |a_1b_4c_5||a_1b_2c_3| = 0 \qquad \text{(C)},$$

already known to us from Monge.

And this is not all, for the next paragraph shows that these two extensions may be repeated in order as often as we please, the opening of the paragraph being as follows (p. 15):—

> “ 17. Généralisons et prouvons que si la formule
>
> $$\begin{pmatrix} k\,l\,..\,q \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,p \\ a\,b\,..\,c \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,p\,..\,q \\ a\,b\,..\,c \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,q \\ a\,b\,..\,c \end{pmatrix}$$
>
> est vraie dans le cas où il y aurait n lettres comprises dans chaque facteur, elle sera encore vraie en ajoutant une nouvelle lettre d dans les seconds facteurs de chaque terme avec l'indice l qui n'y entre pas; et qu'ensuite, si l'on ajoute la même lettre d dans les premiers facteurs de chaque terme avec un nouvel indice r, l'égalité ne sera pas troublée.
>
> Il s'agit donc de démontrer que ces deux formules sont exactes :
>
> $$\begin{pmatrix} k\,l\,..\,q \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,p \\ a\,b\,..\,c\,d \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,p\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix}, \qquad \text{(XXIII. 4)}$$
>
> $$\begin{pmatrix} k\,l\,..\,q\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,p \\ a\,b\,..\,c\,d \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,p\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix}.$$

The line of proof is still the same, and may be shortly indicated by treating the case

$$\text{(D)} \quad |a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| + |a_1b_2c_3||a_1b_2c_4d_5| = 0,$$

which comes immediately after (C), and is derived from it by extending the factors *in which* a_1b_2 *does not occur.* Since by definition

$$|a_1b_2c_3d_4| = d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4| \,,$$

and $$|a_1b_2c_3d_5| = d_5|a_1b_2c_3| - d_3|a_1b_2c_5| + d_2|a_1b_3c_5| - d_1|a_2b_3c_5| \,,$$

it follows that

$$\begin{aligned} &|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| \\ &\quad = \left.\begin{array}{l} \left\{ d_4|a_1b_2c_5| - d_5|a_1b_2c_4| \right\} |a_1b_2c_3| \\ + \left\{ |a_1b_2c_5||a_1b_3c_4| - |a_1b_2c_4||a_1b_3c_5| \right\} d_2 \\ - \left\{ |a_1b_2c_5||a_2b_3c_4| - |a_1b_2c_4||a_2b_3c_5| \right\} d_1 \,. \end{array}\right\} \end{aligned}$$

But the cofactor here of d_2 is by (C) equal to

$$- |a_1b_4c_5||a_1b_2c_3| \,;$$

and the cofactor of d_1

$$= |a_2b_1c_5||a_2b_3c_4| - |a_2b_1c_4||a_2b_3c_5| \,,$$

and therefore by (C)

$$\begin{aligned} &= - |a_2b_1c_3||a_2b_4c_5| \,, \\ &= \quad |a_1b_2c_3||a_2b_4c_5| \,. \end{aligned}$$

Making these substitutions, we have

$$\begin{aligned} &|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5 \\ &= - |a_1b_2c_3| \left\{ d_5|a_1b_2c_4| - d_4|a_1b_2c_5| + d_2|a_1b_4c_5| - d_1|a_2b_4c_5| \right\}, \\ &= - |a_1b_2c_3||a_1b_2c_4d_5| \,, \end{aligned}$$

as was to be shown.

The next three cases are

$$\begin{aligned} |a_1b_2c_5d_6||a_1b_2c_3d_4| - |a_1b_2c_4d_6||a_1b_2c_3d_5| + |a_1b_2c_3d_6||a_1b_2c_4d_5| &= 0 \text{ (E)} \\ |a_1b_2c_3d_4||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6||a_1b_2c_3d_4e_5| &= 0 \text{ (F)} \\ |a_1b_2c_3d_4e_7||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5e_7||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6e_7||a_1b_2c_3d_4e_5| &= 0 \text{ (G)}. \end{aligned}$$

When the factors of each product are of the same order, as in (C), (E), (G) the identity is, in modern phraseology, an "extensional" of (A); that is to say, there is a part common to every factor of the identity, *e.g.*, a_1 in (C), a_1b_2 in (E), $a_1b_2c_3$ in (G), and this common part being deleted, the result is simply the identity

(A). When the factors of each product are of different orders, as in (B), (D), (F), the identity is an "extensional" of something still simpler than (A), viz.,

$$a_1|a_2b_3| - a_2|a_1b_3| + a_3|a_1b_2| = 0\,.$$

In exactly the same manner and at quite as great length the identity

$$\binom{k\ l}{a\ f}\binom{k\ r}{a\ g} - \binom{k\ l}{a\ g}\binom{k\ r}{a\ f} = \binom{k\ l\ r}{a\ f\ g}\binom{k}{a}$$

—already known to us from Lagrange—is made the source of a numerous progeny. By putting figures for k, l, . . . and at the same time writing them as suffixes, these identities, original and derived, take the form

$$|a_1f_2||a_1g_6| - |a_1g_2||a_1f_6| = |a_1f_2g_6||a_1|\,, \qquad (A')$$

$$|a_1f_2||a_1b_2g_6| - |a_1g_2||a_1b_2f_6| = |a_1f_2g_6||a_1b_2|\,, \qquad (B')$$

$$|a_1b_2f_3||a_1b_2g_6| - |a_1b_2g_3||a_1b_2f_6| = |a_1b_2f_3g_6||a_1b_2|\,, \qquad (C')$$

$$|a_1b_2f_3||a_1b_2c_3g_6| - |a_1b_2g_3||a_1b_2c_3f_6| = |a_1b_2f_3g_6||a_1b_2c_3|\,, \qquad (D')$$

$$|a_1b_2c_3f_4||a_1b_2c_3g_6| - |a_1b_2c_3g_4||a_1b_2c_3f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3|\,, \qquad (E')$$

$$|a_1b_2c_3f_4||a_1b_2c_3d_4g_6| - |a_1b_2c_3g_4||a_1b_2c_3d_4f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3d_4|\,, \qquad (F')$$

$$|a_1b_2c_3d_4f_5||a_1b_2c_3d_4g_6| - |a_1b_2c_3d_4g_5||a_1b_2c_3d_4f_6| = |a_1b_2c_3d_4f_5g_6||a_1b_2c_3d_4|\,. \qquad (G')$$

Of these (C′), (E′), (G′) deserve to be noted, being along with the original (A′) extensionals of the manifest identity

$$f_2g_6 - g_2f_6 = |f_2g_6|\,. \qquad (\text{XXIII. } 5).$$

On the other hand (B′), (D′), (F′) are essentially the same as (B), (D), (F) already obtained—a fact which Desnanot overlooks.

As the source of a third series of results, obtained in still the same way, the identity

$$\binom{k\,l}{a\,h}\binom{k\,l}{f\,g} - \binom{k\ l}{a\ g}\binom{k\,l}{f\,h} = \binom{k\,l}{a\,f}\binom{k\,l}{h\,g} \quad . \quad . \quad (A'')$$

is next taken. In reality, however, this does not differ from the first identity so treated, viz.,

$$\binom{k\ l}{a\ b}\binom{m\,p}{a\ b} - \binom{k\ p}{a\ b}\binom{m\ l}{a\ b} = \binom{k\ m}{a\ b}\binom{l\ p}{a\ b} \quad . \quad . \quad (A).$$

In (A) the letters *ab* remain unchanged throughout, and the indices vary; while in (A″) the indices remain the same, and the letters vary. As we should now say, the difference is a mere matter of rows and columns. The derived identities (B″), (C″), (D″), . . . are consequently found to be quite the same as (B), (C), (D), ...

The fourth and last source made use of is the well-known theorem regarding the aggregate of products whose first factors constitute what Cauchy would have called a "suite verticale," and whose second factors are the cofactors, in the determinant of the system, of another "suite verticale." Desnanot however, viewing the theorem from a different stand-point, enunciates it as follows (p. 26):—

> "*Si l'on a* n *lettres* ab .... cdf, *et qu'on les combine* n – 1 *à* n – 1, *on aura* n *arrangemens* ab .... cd, ab .... cf, ab .... df, ... , a .... cdf, b .... cdf; *qu'on applique dans chaque arrangement les* n – 1 *indices* kl....mp, *ce qui donnera ces quantités*

$$\left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;c\;d\end{matrix}\right),\ \left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;c\;f\end{matrix}\right),\ \left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;d\;f\end{matrix}\right),\ \cdots\cdots \left(\begin{matrix}k\;l\;..\;m\;p\\ a\;..\;c\;d\;f\end{matrix}\right),\ \left(\begin{matrix}k\;l\;..\;m\;p\\ b\;..\;c\;d\;f\end{matrix}\right);$$

> *et qu'ensuite on les multiplie chacune par la lettre qui n'entre pas dans l'arrangement en l'affectant d'un même indice et donnant au produit le signe* plus *ou le signe* moins, *suivant que la lettre multiplicateur occupe un rang impair ou pair dans les* n *lettres, en partant de la droite, la somme des produits sera zéro.*" (xii. 9).

Before proceeding to deduce others from it, he gives a proof of it for the case

$$(\mathrm{B}''')\quad \overset{p}{f}\left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;c\;d\end{matrix}\right) - \overset{p}{d}\left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;c\;f\end{matrix}\right) + \overset{p}{c}\left(\begin{matrix}k\;l\;..\;m\;p\\ a\;b\;..\;d\;f\end{matrix}\right) - \cdot\ \cdot\ \cdot\ \cdot$$

$$\mp\ \overset{p}{b}\left(\begin{matrix}k\;l\;..\;m\;p\\ a\;..\;c\;d\;f\end{matrix}\right) \pm \overset{p}{a}\left(\begin{matrix}k\;l\;..\;m\;p\\ b\;..\;c\;d\;f\end{matrix}\right) = 0\,.$$

The method of proof is interesting, because it depends almost entirely on the definition which Desnanot follows Vandermonde in using. It will be readily understood by seeing it applied to the simple case

$$b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4\,|b_1c_2d_3| = 0\,.$$

Expanding each of the determinants $|b_2c_3d_4|$, $|b_1c_3d_4|$, ... in terms of the b's and their cofactors, we have

$$\begin{array}{rl}
 & b_1|b_2c_3d_4| \quad -\, b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4|b_1c_2d_3| \\
= & \left.\begin{array}{rlllll}
b_1\{ & & b_2|c_3d_4| & -\,b_3|c_2d_4| & +\,b_4|c_2d_3| & \} \\
-\,b_2\{ & b_1|c_3d_4| & & -\,b_3|c_1d_4| & +\,b_4|c_1d_3| & \} \\
+\,b_3\{ & b_1|c_2d_4| & -\,b_2|c_1d_4| & & +\,b_4|c_1d_2| & \} \\
-\,b_4\{ & b_1|c_2d_3| & +\,b_2|c_1d_3| & +\,b_3|c_1d_2| & & \}
\end{array}\right\},
\end{array}$$

$$=0,$$

for the terms in the expanded form destroy each other in pairs.

The derived identities are obtained exactly in the manner followed by Bézout in 1779 (see pp. 457–8, Vol. XIV.). The fundamental identity is taken, say in the form

$$f_5|a_1b_2c_3d_4e_5| - e_5|a_1b_2c_3d_4f_5| + d_5|a_1b_2c_3e_4f_5| - c_5|a_1b_2d_3e_4f_5| \\ + b_5|a_1c_2d_3e_4f_5| - a_5|b_1c_2d_3e_4f_5| = 0,$$

and another instance is put alongside of it, in which the same letters and suffixes are involved, say

$$f_1|a_1b_2c_3d_4e_5| - e_1|a_1b_2c_3d_4f_5| + d_1|a_1b_2c_3e_4f_5| - c_1|a_1b_2d_3e_4f_5| \\ + b_1|a_1c_2d_3e_4f_5| - a_1|b_1c_2d_3e_4f_5| = 0.$$

One of the constituent determinants, say the last, $|b_1c_2d_3e_4f_5|$ is then eliminated by equalisation of coefficients and subtraction, the result being

$$|a_1f_5| \cdot |a_1b_2c_3d_4e_5| - |a_1e_5||a_1b_2c_3d_4f_5| + |a_1d_5||a_1b_2c_3e_4f_5| \\ + |a_1c_5||a_1b_2d_3e_4f_5| - |a_1b_5||a_1c_2d_3e_4f_5| = 0 \quad (\mathrm{C}''')$$

In the next place, two additional instances of this derived identity are taken along with it, the first differing from it in having a 2 instead of a 5 in all the first factors, and the second in having a 2 instead of a 1; viz.,

$$|a_1f_2||a_1b_2c_3d_4e_5| - |a_1e_2||a_1b_2c_3d_4f_5| + |a_1d_2||a_1b_2c_3e_4f_5| \\ + |a_1c_2||a_1b_2d_3e_4f_5| - |a_1b_2||a_1c_2d_3e_4f_5| = 0,$$

and

$$|a_2f_5||a_1b_2c_3d_4e_5| - |a_2e_5||a_1b_2c_3d_4f_5| + |a_2d_5||a_1b_2c_3e_4f_5| \\ + |a_2c_5||a_1b_2d_3e_4f_5| - |a_2b_5||a_1c_2d_3e_4f_5| = 0.$$

Multiplication by $b_2, -b_5, -b_1$ is then effected and addition performed, when by reason of such identities as

$$b_2|a_1f_5| - b_5|a_1f_2| - b_1|a_2f_5| = |a_1b_2f_5|,$$

and $$b_2|a_1b_5| - b_5|a_1b_2| - b_1|a_2b_5| = 0,$$

elimination of $|a_1c_2d_3e_4f_5|$ is produced, and the result takes the form

$$(\mathrm{D}''') \quad \begin{aligned} a_1b_2f_5||a_1b_2c_3d_4e_5| - |a_1b_2e_5||a_1b_2c_3d_4f_5| + |a_1b_2d_5||a_1b_2c_3e_4f_5| \\ - a_1b_2c_5||a_1b_2d_3e_4f_5| = 0. \end{aligned}$$

The process of derivation may be pursued further, giving next an identity in which the first factors are all of the fourth order. Desnanot says (pp. 31, 32)—

> "Pour ne pas nous répéter constamment, nous dirons que cette formule s'étendrait à un nombre quelconque de lettres placées dans les premiers facteurs, et que
>
> $$(\mathrm{H}''') \quad \begin{aligned} \begin{pmatrix} k\,l\,.\,.\,.\,p \\ a\,b\,.\,.\,.\,f \end{pmatrix}\begin{pmatrix} k\,l\,.\,.\,.\,m\,p \\ a\,b\,.\,.\,.\,c\,d \end{pmatrix} - \begin{pmatrix} k\,l\,.\,.\,.\,p \\ a\,b\,.\,.\,.\,d \end{pmatrix}\begin{pmatrix} k\,l\,.\,.\,.\,m\,p \\ a\,b\,.\,.\,.\,c\,f \end{pmatrix} \\ + \begin{pmatrix} k\,l\,.\,.\,.\,p \\ a\,b\,.\,.\,.\,c \end{pmatrix}\begin{pmatrix} k\,l\,.\,.\,.\,m\,p \\ a\,b\,.\,.\,.\,d\,f \end{pmatrix} - \;.\;\;.\;\;.\;\;.\;\;. = 0. \end{aligned}$$
>
> Les termes sont alternativement positifs et négatifs, les indices sont les mêmes dans les premiers facteurs de chaque terme, ils font partie des indices qui se trouvent dans les autres facteurs et sont placés dans le même ordre; quant aux lettres, il y a ou une, ou deux, ou trois, etc. lettres communes aux seconds facteurs écrites toujours dans le même ordre et suivies de la $n^{\text{ième}}$ lettre qui n'entre pas dans les seconds facteurs; de sorte que s'il y a n' lettres communes à tous les facteurs, le nombre des termes de (H''') sera $n - n'$." (XXIII. 6)

The general result (H''') is simply what would now be called the extensional of the identity of Vandermonde from which Desnanot derives it.

Co-ordinate, in a sense, with the said identity, is that other which Desnanot uses as a definition; and this latter is the next of which the extensional is found. The process, so far as indicated, is exactly similar to that employed in the preceding case. The results obtained are

$$
\text{(B'''')}\quad
\begin{aligned}
&\binom{p\,r}{a\,f}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,d} - \binom{p\,r}{a\,d}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,f} + \binom{p\,r}{a\,c}\binom{k\,l\ldots m\,p}{a\,b\ldots d\,f}\\
&\quad\cdot\;\cdot\;\cdot\;\cdot \mp \binom{p\,r}{a\,b}\binom{k\,l\ldots m\,p}{a\ldots c\,d\,f} = \binom{p}{a}\binom{k\,l\ldots m\,p\,r}{a\,b\ldots c\,d\,f}
\end{aligned}
$$

$$
\text{(C'''')}\quad
\begin{aligned}
&\binom{k\,p\,r}{a\,b\,f}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,d} - \binom{k\,p\,r}{a\,b\,d}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,f}\\
&+\binom{k\,p\,r}{a\,b\,c}\binom{k\,l\ldots m\,p}{a\,b\ldots d\,f} - \cdots\cdots = \binom{k\,p}{a\,b}\binom{k\,l\ldots m\,p\,r}{a\,b\ldots c\,d\,f};
\end{aligned}
$$

and the general result including them is referred to. (XLVI.)

That they are extensionals of the definition is evident from the fact that the index p may be moved to the left so as to make $\overset{p}{a}$ common to every factor of (B''''), and $\overset{k}{a}\,\overset{p}{b}$ common to every factor of (C'''').

Still another series of results is obtained, but they are essentially the same as the foregoing, the difference again being merely a matter of rows and columns.

All these preparations having been made, Desnanot returns to the subject of the relations between the numerators and denominators of the values of the unknowns in a set of linear equations. Thirteen pages are occupied with the case of four unknowns, the number of relations found being 74, of which, after scrutiny, 14 are retained. The case of five unknowns, and the case of six unknowns are gone into with about as much detail, and then, lastly, the general set of n equations with n unknowns is dealt with. None of the relations obtained need be given, as they are all included in the identities which have been spoken of above as extensionals.

The second chapter (p. 94) bears the heading

> *Simplification des formules générales qui donnent les valeurs des inconnues dans les équations du premier degré, lorsqu'on veut les évaleur en nombres.*

Here again the cases of three, four, five, six unknowns are dwelt upon with equal fulness in succession. The consideration of one of them will suffice to show the nature of the method, and will enable the reader to judge of the amount of labour saved by employing it. Choosing the case of four unknowns, we find at the outset the equations stated and the solution condensed as follows (p. 104):—

"EQUATIONS DONNÉES.

$$\begin{aligned} ax &+ by + cz + dt = f \\ a'x &+ b'y + c'z + d't = f' \\ a''x &+ b''y + c''z + d''t = f'' \\ a'''x &+ b'''y + c'''z + d'''t = f'''. \end{aligned}$$

CALCUL.

$$\begin{aligned} ab' - ba' &= \alpha , \\ ab'' - b''a &= \beta , \\ a'b'' - b''a' &= \gamma , \\ ab''' - ba''' &= \delta , \\ a'b''' - b'a''' &= \epsilon ; \end{aligned}$$

$$\begin{aligned} m &= c''\alpha - c'\beta + c\gamma , \\ n &= c'''\alpha - c'\delta + c\epsilon , \\ m' &= f''\alpha - f'\beta + f\gamma , \\ n' &= f'''\alpha - f'\delta + f\epsilon ; \end{aligned}$$

$$D = \frac{1}{\alpha} \left\{ m(\alpha d''' - \delta d' + \epsilon d) - n(\alpha d'' - \beta d' + \gamma \delta) \right\} .$$

$$N''' = \frac{1}{\alpha} \left(mn' - nm' \right) ,$$

$$N'' = \frac{1}{\alpha} \left\{ m'(\alpha d''' - \delta d' + \epsilon d) - n'(\alpha d'' - \beta d' + \gamma d) \right\} ,$$

$$\begin{aligned} f\,D - cN'' - dN''' &= S , \\ f'D - c'N'' - d'N''' &= S' , \end{aligned}$$

$$N' = \frac{aS' - Sa'}{\alpha} ,$$

$$N = \frac{Sb' - bS'}{\alpha} ,$$

$$x = \frac{N}{D} , \quad y = \frac{N'}{D} , \quad z = \frac{N''}{D} , \quad t = \frac{N'''}{D} ."$$

The explanation of the mode of procedure is not difficult to see.

(1) The determinants $|ab'|$, $|ab''|$, $|a'b''|$, $|ab'''|$, $|a'b'''$ are calculated.

(2) With the help of these are next got four of a higher order, viz. $|ab'c''|$, $|ab'c'''|$, $|ab'f''|$, $|ab'f'''|$.

(3) Two others of the same order, viz.

$$\begin{array}{lcc} & \alpha d''' - \delta d' + \epsilon d , & \alpha d'' - \beta d' + \gamma d \\ \textit{i.e.} & & \\ & |ab'd'''| , & |ab'd''| , \end{array}$$

having been calculated, the identity

$$|ab'|\,.\,\mathrm{D} = |ab'c''|\,.\,|ab'd'''| - |ab'c'''|\,.\,|ab'd''|$$

is used to find D.

(4) A similar identity

$$|ab'|\,.\,\mathrm{N}''' = |ab'c''|\,.\,|ab'f'''| - |ab'c'''|\,.\,|ab'f''|$$

is used to find N‴.

(5) A similar identity

$$|ab'|\,.\,\mathrm{N}'' = |ab'f''|\,.\,|ab'd'''| - |ab'f'''|\,.\,|ab'd''|$$

is used to find N″.

(6) Two subsidiary quantities S, S′ are calculated, the first being

$$= f|ab'c''d'''| - c|ab'f''d'''| - d|ab'c''f'''|\,,$$

and the second

$$= f'\,|ab'c''d'''| - c'\,|ab'f''d'''| - d'|ab'c''f'''|\,.$$

(7) From these N′ and N are readily got. For evidently

$$a\mathrm{S}' - \mathrm{S}a'$$

$$= |af'|\,.\,|ab'c''d'''| - |ac'|\,.\,|ab'f''d'''| - |ad'|\,.\,|ab'c''f'''|$$

and this by a previous theorem

$$= |ab'|\,.\,|af'c''d'''|\,,$$

$$= |ab'|\,.\,\mathrm{N}'\,. \qquad \text{(XIII. 3.)}$$

The third chapter consists of a lengthy examination (pp. 157–264) of the singular cases met with in the solution of linear equations, and does not at present concern us.

CAUCHY (1821).

[Cours d'Analyse de l'Ecole Royale Polytechnique I. xvi. + 576 pp. Paris.]

When Cauchy came to write his *Course of Analysis*, afterwards so well known, he did not fail to assign a position in it to the subject of his memoir of 1812. The third chapter bears the heading, "*Des Fonctions Symétriques et des Fonctions Alternées.*"

It occupies, however, only fifteen pages (pp. 70–84), and of these only nine are devoted to alternating functions and the solution of simultaneous linear equations. Of course, in so limited a space, the merest sketch of a theory is all that is possible. An alternating function is first defined, the word "alternée" being now set in contrast with "symétrique," and not, as formerly, with "permanente." Functions other than those that are rational and integral being left aside, the latter, if alternating, are shown (1) to consist of as many positive as negative terms, in each of which all the variables occur with different indices, and (2) to be divisible by the simplest of all alternating functions of the variables, viz., the difference-product. The set of equations

$$a_r x + b_r y + c_r z + \ldots + g_r u + h_r v = k_r \qquad (r = 0, 1, \ldots, n-1)$$

is then attacked, the method being—to take the difference-product of $a, b, \ldots h$,—denote by D what the expansion of this becomes when exponents are changed into suffixes,—denote by A_r the co-factor of a_r in D,—then obtain the equations

$$\begin{aligned} A_0a_0 + A_1a_1 + A_2a_2 + \ldots + A_{n-1}a_{n-1} &= D\,, \\ A_0b_0 + A_1b_1 + A_2b_2 + \ldots + A_{n-1}b_{n-1} &= 0\,, \\ A_0c_0 + A_1c_1 + A_2c_2 + \ldots + A_{n-1}c_{n-1} &= 0\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ A_0h_0 + A_1h_1 + A_2h_2 + \ldots + A_{n-1}h_{n-1} &= 0\,, \end{aligned}$$

—and thereafter proceed as Laplace had taught. As in the memoir of 1812, the "symbolic" form of the values of $x, y, \ldots$ is unfailingly given.

A note is added (pp. 521–524) on the development of the difference-product, showing how all the terms may be got from one by interchanging one exponent with another, how the signs depend on the number of said interchanges, and how it may be ascertained whether any two given terms have like or unlike signs.

It will thus be seen that not only is the name "determinant" never mentioned in the chapter, and the notation $S \pm a_0b_1c_2 \ldots h_{n-1}$ never used, but that the subject is scarcely so much as touched upon. Although, therefore, Cauchy's text-book went through a considerable number of editions, and had a widespread influence, it gave no such impulse as it might have done to the study of the theory of determinants.

SCHERK (1825).

[Mathematische Abhandlungen. Von Dr. Heinrich Ferdinand Scherk, . . . iv. + 116 pp. Berlin. Pp. 31–66. Zweite Abhandlung: Allgemeine Auflösung der Gleichungen des ersten Grades mit jeder beliebigen Anzahl von unbekannten Grössen, und einige dahin gehörige analytische Untersuchungen.]

The only previous writings of importance known to Scherk were, according to his own statement, those of Cramer, Bézout (1764), Vandermonde, Bézout (1779), Hindenburg, and Rothe. His style bears most resemblance to Rothe's, whose paper, however, he does not speak of with unmixed eulogy, characterising it as containing "eine strenge aber ziemlich weitläuftige Auflösung der Aufgabe."

The main part of the memoir consists of a lengthy demonstration, extending, indeed, to 17 pages quarto, of Cramer's rule, or rather of Cramer's set of three rules (iv., v., iii. 2), by the method of so-called mathematical induction. The peculiarity of the demonstration is that it is entered upon without any previous examination of the properties of Cramer's functions (determinants); and it is noteworthy on two grounds—(1) as being new, and (2) because the properties, which it really if not explicitly employs, had also not been previously referred to.

The cases of one equation with one unknown, two equations with two unknowns, three equations with three unknowns, are dealt with in succession, the solution of one case being used in obtaining the solution of the next. All three solutions are noted as being in accordance with Cramer's rules, and the said rules being formulated, and supposed to hold,—for n equations with n unknowns, it is sought to establish their validity for $n+1$ equations with $n+1$ unknowns. In other words, the set of n equations being

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{1}{a}}\overset{n-2}{x} + \dots\dots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{2}{a}}\overset{n-2}{x} + \dots\dots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{n}{a}}\overset{n-2}{x} + \dots\dots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s}
\end{array}\right\}$$

and the corresponding values of

$$\overset{1}{x},\quad \overset{2}{x},\quad \ldots\ldots\quad \overset{n}{x},$$

being

$$\frac{P\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\,\overset{1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)},\quad \frac{P\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{2}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\,\overset{2}{\underset{h}{a}}\,,\,\overset{2}{\underset{h}{a}}\right)},\quad \ldots\ldots,\quad \frac{P\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n}{\underset{h}{a}}\,,\,\overset{n}{\underset{h}{a}}\right)},$$

it is required to show that the solution of the set of $n+1$ equations

$$\left.\begin{array}{l}
\overset{n+1}{\underset{1}{a}}\overset{n+1}{x} + \overset{n}{\underset{1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n+1}{\underset{2}{a}}\overset{n+1}{x} + \overset{n}{\underset{2}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{n+1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n+1}{a}}\overset{1}{x} = \underset{n+1}{s}
\end{array}\right\}$$

is

$$\overset{1}{x} = \frac{P\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\overset{1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)},\quad \ldots\ldots,\quad \overset{n+1}{x} = \frac{\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\underset{h}{s}\,,\,\overset{n+1}{\underset{h}{a}}\right)}{\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{n+1}{\underset{h}{a}}\right)}. \qquad \text{(XIII. 4.)}$$

Before proceeding, the notation

$$P\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)$$

requires attention. It is meant to be an epitome of Cramer's rules; the first half of the group of symbols, viz. $P\Big(\overset{n}{\underset{n}{a}}$ implying permutation of the under-indices of the product $\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\ldots\overset{n}{\underset{n}{a}}$ and aggregation of the different products thus obtained, each taken with its proper sign: and the second half implying that in every term of this aggregate $\underset{h}{s}$ is to be substituted for $\underset{h}{a}$. A modern writer would denote the same thing by

$$\begin{vmatrix}
\underset{1}{s} & \overset{2}{\underset{1}{a}} & \overset{3}{\underset{1}{a}} & \ldots & \overset{n}{\underset{1}{a}} \\
\underset{2}{s} & \overset{2}{\underset{2}{a}} & \overset{3}{\underset{2}{a}} & \ldots & \overset{n}{\underset{2}{a}} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\underset{n}{s} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \ldots & \overset{n}{\underset{n}{a}}
\end{vmatrix},$$

only it must be noted that in using $\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s}\,,\ \overset{1}{\underset{h}{a}}\right)$ a this stage, we leave out of account the signs of the terms composing it, the rule of signs being the subject of a separate investigation. Any one of the forms

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{1}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right),\quad \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{2}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right),\ \ldots\ldots$$

it need scarcely be added, will thus stand for the common denominator.

Of the $n+1$ equations the first n are taken, written in the form

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\ \overset{n-1}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} - \overset{n+1}{\underset{1}{a}}\ \overset{n+1}{x} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\ \overset{n-1}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} - \overset{n+1}{\underset{2}{a}}\ \overset{n+1}{x} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\ \overset{n-1}{x} + \ldots + \overset{k}{\underset{n}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s} - \overset{n+1}{\underset{n}{a}}\ \overset{n+1}{x}
\end{array}\right\}$$

and solved, the results being by hypothesis

$$\overset{1}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\ \overset{n+1}{x}\,,\ \overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{1}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right)},$$

$$\ldots\ldots\ldots\ldots$$

$$\overset{k}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\ \overset{n+1}{x}\,,\ \overset{k}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{2}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right)},$$

$$\ldots\ldots\ldots\ldots$$

$$\overset{n}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\ \overset{n+1}{x}\,,\ \overset{n}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{n}{\underset{h}{a}}\,,\ \overset{n}{\underset{h}{a}}\right)}.$$

These values are then of course substituted in the $(n+1)^{\text{th}}$ equation, which thus becomes

$$a^{n+1}_{n+1} x^{n+1} + a^{n}_{n+1} \frac{P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{1}_{h}\right)}{P\left(a^{n}_{n} ; a^{1}_{h} , a^{1}_{h}\right)} + \ldots + a^{k}_{n+1} \frac{P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{k}_{h}\right)}{P\left(a^{n}_{n} ; a^{k}_{h} , a^{k}_{h}\right)}$$

$$+ \ldots + a^{1}_{n+1} \frac{P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{1}_{h}\right)}{P\left(a^{n}_{n} ; a^{n}_{h} , a^{n}_{h}\right)} = s_{n+1} ;$$

and as this manifestly involves none of the unknowns but x^{n+1}, the object must now be to solve for x^{n+1}, and then show what the value obtained is transformable into. The way in which this is effected is well worthy of attention. Scherk's own words in regard to the first steps are (p. 40)—

> "Da aber $s_{h} - a^{n+1}_{h} x^{n+1}$ in jeder einzelnen Permutationsform nur Einmal, nämlich in der ersten Potenz vorkommt, so bedeutet das Zeichen
>
> $$P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{k}_{h}\right)$$
>
> dass in jede der in I. beschriebenen Permutationsformen für a^{k}_{h} erst s_{h}, dann $a^{n+1}_{h} x^{n+1}$ gesetzt, und beide Resultate von einander abgezogen werden sollen: folglich ist
>
> $$P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{k}_{h}\right) = P\left(a^{n}_{n} ; s_{h} , a^{k}_{h}\right) - P\left(a^{n}_{n} ; a^{n+1}_{h} x^{n+1} , a^{k}_{h}\right) .$$
>
> In dem letzten Gliede dieser Gleichung kömmt aber in jeder Form x^{n+1}, und zwar zur ersten Potenz, vor; x^{n+1} ist also gemeinschaftlicher Factor aller Formen, und folglich ist
>
> $$P\left(a^{n}_{n} ; s_{h} - a^{n+1}_{h} x^{n+1} , a^{k}_{h}\right) = P\left(a^{n}_{h} ; s_{h} , a^{k}_{h}\right) - x^{n+1} P\left(a^{n}_{n} ; a^{n+1}_{h} , a^{k}_{h}\right) .$$
>
> Macht man diese Substitution für $k = 1, 2, \ldots, n$, in der letzten Gleichung, und bemerkt, dass
>
> $$P\left(a^{n}_{n} ; a^{1}_{h} , a^{1}_{h}\right) = P\left(a^{n}_{n} ; a^{2}_{h} , a^{2}_{h}\right) = \ldots . = P\left(a^{n}_{n} ; a^{k}_{h} , a^{k}_{h}\right) ,$$

so geht diese in folgende Gleichung über

$$\overset{n+1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right) \overset{n+1}{x}$$

$$+ \left\{ \overset{n}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{n}{\underset{h}{a}}\right) + \ldots + \overset{k}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{k}{\underset{h}{a}}\right) + \ldots + \overset{1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{1}{a}\right) \right\}$$

$$- \left\{ \overset{n}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{n}{\underset{h}{a}}\right) + \ldots + \overset{k}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right) + \ldots + \overset{1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{1}{\underset{h}{a}}\right) \right\} \overset{n+1}{x}$$

$$= \underset{n+1}{s}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right) ;$$

folglich

$$\overset{n+1}{x} = \frac{- \overset{1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{1}{\underset{h}{a}}\right) - \overset{2}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{2}{\underset{h}{a}}\right) - \ldots - \overset{n}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{n}{\underset{h}{a}}\right) + \underset{n+1}{s}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right)}{- \overset{1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{1}{\underset{h}{a}}\right) - \overset{2}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{2}{\underset{h}{a}}\right) - \ldots - \overset{n}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{n}{\underset{h}{a}}\right) + \overset{n+1}{\underset{n+1}{a}}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right)} .$$

The first theorem here made use of and formulated, viz.,

$$P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}} \overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right) = P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{k}{\underset{h}{a}}\right) - P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} \overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right) \qquad \text{(XLVII.)}$$

is the now familiar rule for the partition of a determinant with a row or column of binomial elements into two determinants, or for the addition of two determinants which are identical except in one row or one column. The second theorem, viz.,

$$P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} \overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right) = \overset{n+1}{x}\, P\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right) \qquad \text{(XLVIII.)}$$

is the now equally familiar theorem regarding the multiplication of a determinant by means of the multiplication of all the elements of a row or column. That these two very elementary theorems should not have been noted until the time of Scherk is rather remarkable.

The consideration of the constitution of

$$P\left(\overset{n+1}{\underset{n+1}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right)$$

is next entered upon, with the object of showing that the terms are exactly the terms of the denominator

$$-\overset{1}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{n+1}{\underset{h}{a}}\ ,\ \overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{n+1}{\underset{h}{a}}\ ,\ \overset{2}{\underset{h}{a}}\right)-\ldots\ldots+\overset{n+1}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{k}{\underset{h}{a}}\ ,\ \overset{k}{\underset{h}{a}}\right).$$

More than two pages are occupied with this part proof of Bézout's recurrent law of formation. The identity of the terms of

$$P\left(\overset{n+1}{\underset{n+1}{a}}\ ;\ \underset{h}{s}\ ,\ \overset{n+1}{\underset{h}{a}}\right)$$

with the terms of the numerator then follows at once; and the desired form for the value of $\overset{n+1}{x}$, so far as the *magnitude* of the terms is concerned, is thus obtained. The corresponding forms for $x_1, x_2, \ldots$ are of course immediately deducible.

The rules for obtaining the terms of the numerator and denominator having been thus established in all their generality, the rule of signs is next dealt with. The treatment is cumbersome, but fresh and interesting. It is pointed out, to start with, that the counting of the inversions of order of a permutation, is equivalent to subtracting separately from each element all the elements which follow it, reckoning $+1$ as a sign-factor when the difference is positive, and -1 when the difference is negative, and then taking the product of all the said factors. This, it will be recalled, is essentially identical with an observation of Cauchy's. Scherk, however, goes on to remark that these sign-factors may be viewed as functions of the differences which give rise to them, and may be so represented. Whether there actually be a function which equals $+1$ for all positive values of the argument and equals -1 for all negative values is left for future consideration. Cramer's rule of signs is thus made to take the following form (p. 45):—

> "Wenn $\phi(\beta)$ eine solche Function der ganzen Zahl β ist, welche $= +1$ ist, wenn β positiv, und -1, wenn β negativ ist, so ist das Vorzeichen Z irgend eines in dem Aggregate
>
> $P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{h}{\underset{k}{a}}\ ,\ \overset{h}{\underset{k}{a}}\right)$ enthaltenen Gliedes
>
> $$\overset{1}{\underset{\alpha}{a}}\,\overset{2}{\underset{\alpha}{a}}\,\overset{3}{\underset{\alpha'''}{a}}\ldots\overset{k-1}{\underset{\alpha^{(k-1)}}{a}}\ \overset{k}{\underset{\alpha^{(k)}}{a}}\ \overset{k-1}{\underset{\alpha^{(k+1)}}{a}}\ \ldots\overset{n}{\underset{\alpha^{(n)}}{a}}$$
>
> folgendes:

$$
\begin{aligned}
Z = {} & \phi(\alpha'' - \alpha').\phi(\alpha''' - \alpha')\ldots\phi(\alpha^{(k-1)} - \alpha').\phi(\alpha^{(k)} - \alpha').\phi(\alpha^{(k+1)} - \alpha')\ldots\phi(\alpha^{(n)} - \alpha') \\
& \phi(\alpha''' - \alpha'').\phi(\alpha'''' - \alpha'')\ldots\phi(\alpha^{(k)} - \alpha'').\phi(\alpha^{(k+1)} - \alpha'').\phi(\alpha^{(k+2)} - \alpha'')\ldots\phi(\alpha^{(n)} - \alpha'') \\
& \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
& \phi(\alpha^{(k-1)} - \alpha^{(k-2)}).\phi(\alpha^{(k)} - \alpha^{(k-2)})\ldots\phi(\alpha^{(n)} - \alpha^{(k-2)}) \\
& \phi(\alpha^{(k)} - \alpha^{(k-1)}).\phi(\alpha^{(k+1)} - \alpha^{(k-1)})\ldots\phi(\alpha^{(n)} - \alpha^{(k-1)}) \\
& \phi(\alpha^{(k+1)} - \alpha^{(k)}).\phi(\alpha^{(k+2)} - \alpha^{(k)})\ldots\phi(\alpha^{(n)} - \alpha^{(k)}) \\
& \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
& \phi(\alpha^{(n)} - \alpha^{(n-1)}). \qquad\qquad (\text{III. } 22.)
\end{aligned}
$$

And it is this form which Scherk seeks to establish. The mode of proof is again the so-called inductive mode. In the case of *two* permutable indices the law is readily seen to hold. We thus have, preparatory for the next case,

$$\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\,,$$

$$\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\,,$$

$$\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\,.$$

But "nach dem Obigem"

$$\mathrm{P}\left(\overset{3}{\underset{3}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\right) = -\overset{1}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right) - \overset{2}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right) + \overset{3}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\right).$$

Consequently

$$
\begin{aligned}
\mathrm{P}\left(\overset{3}{\underset{3}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\right) = {} & -\overset{1}{\underset{3}{a}}\left\{\phi(2-1)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\right\} \\
& -\overset{2}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\right\} \\
& -\overset{3}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\right\}.
\end{aligned}
$$

But as

$$
\begin{aligned}
-\phi(2-1) &= \phi(1-2) \\
\text{and} \qquad -1 &= \phi(2-3) \\
\text{and} \qquad -1 &= \phi(1-3)\,,
\end{aligned}
$$

we may substitute

$$\phi(1-2)\phi(2-3)\phi(1-3) \quad \text{for} \quad -\phi(2-1).$$

This and five other similar substitutions give us

$$\begin{aligned}P\left(\overset{3}{\underset{3}{a}};\overset{k}{\underset{h}{a}},\overset{k}{\underset{h}{a}}\right)=&\ \phi(2-1)\phi(3-1)\phi(3-2)\ \overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}+\phi(3-1)\phi(2-1)\phi(2-3)\ \overset{1}{\underset{1}{a}}\overset{2}{\underset{3}{a}}\overset{3}{\underset{2}{a}}\\&+\phi(1-2)\phi(3-2)\phi(3-1)\ \overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\overset{3}{\underset{3}{a}}+\phi(3-2)\phi(1-2)\phi(1-3)\ \overset{1}{\underset{2}{a}}\overset{2}{\underset{3}{a}}\overset{3}{\underset{1}{a}}\\&+\phi(1-3)\phi(2-3)\phi(2-1)\ \overset{1}{\underset{3}{a}}\overset{2}{\underset{1}{a}}\overset{3}{\underset{2}{a}}+\phi(2-3)\phi(1-3)\phi(1-2)\ \overset{1}{\underset{3}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\ ;\end{aligned}$$

so that the law is seen to hold also for the case of *three* permutable indices. The completion of the proof, giving the transition from n to $n+1$ permutable indices, occupies three pages.

This is followed by two pages devoted to the subjects temporarily set aside at the outset, viz., the possible existence of functions having the peculiar properties of ϕ. Two amusing instances of such functions are given,—

$$(1)\qquad \left.\begin{aligned}\phi(\beta)=&\ P_0^{\beta-1}+P_0^{\beta-2}+P_0^{\beta-3}+\ \ldots\\&-P_0^{\beta-1}-P_0^{\beta-2}-P_0^{\beta-3}-\ \ldots\end{aligned}\right\}$$

$$(2)\qquad \left.\begin{aligned}\phi(\beta)=&\ \frac{\sin 2\beta\pi}{(\beta-1)2\pi}+\frac{\sin 2\beta\pi}{(\beta-2)2\pi}+\frac{\sin 2\beta\pi}{(\beta-3)2\pi}+\ldots\\&-\frac{\sin 2\beta\pi}{(\beta+1)2\pi}-\frac{\sin 2\beta\pi}{(\beta+2)2\pi}-\frac{\sin 2\beta\pi}{(\beta+3)2\pi}-\ldots\end{aligned}\right\}$$

$$=\left(\frac{1}{\beta^2-1}+\frac{2}{\beta^2-4}+\frac{3}{\beta^2-9}+\ldots\right)\frac{\sin 2\beta\pi}{\pi}.$$

where P_0^k stands for the k^{th} coefficient in the expansion of $(a+b)^0$. Success, however far from brilliant it may be, in thus expressing the rule of signs by means of the symbols of analyses, led Scherk to try to do the same for the rule of formation of the terms. Nothing came of the attempt, however. "Bald aber," he says, zeigte es sich dass Permutationen niemals durch andere analytische Zeichen ersetzt werden könnten."

Such speculations are not altogether uninteresting when later work like Hankel's comes to be considered.

In an Appendix dealing (1) with the case of a set of linear equations which are not all independent, (2) with the solution of particular sets of equations, there is given at the outset a proof of the theorem regarding the sign of a permutation which is got from another permutation by the interchange of two elements. If the under-indices of the one term whose sign is z be

$$a'a''a''' \dots a^{(i-1)}a^{(i)}a^{(i+1)} \dots\dots a^{(k-1)}a^{(k)}a^{(k+1)} \dots a^{(n)},$$

and of the other whose sign is Z * be

$$a'a''a''' \dots a^{(i-1)}a^{(k)}a^{(i+1)} \dots a^{(k-1)}a^{i}a^{(k+1)} \dots a^{(n)}$$

it is shown that

$$\begin{aligned}\frac{z}{Z} &= \frac{\phi(a^{i+1}-a^{i})}{\phi(a^{i}-a^{i+1})} \cdot \frac{\phi(a^{i+2}-a^{i})}{\phi(a^{i}-a^{i+2})} \cdots \frac{\phi(a^{k}-a^{i})}{\phi(a^{i}-a^{k})} \\ &\times \frac{\phi(a^{k}-a^{i})}{\phi(a^{i}-a^{k})} \cdot \frac{\phi(a^{k}-a^{i+1})}{\phi(a^{i+1}-a^{k})} \cdots \frac{\phi(a^{k}-a^{k-1})}{\phi(a^{k-1}-a^{k})};\end{aligned}$$

and there being here $2k-2i-1$ quotients each $= -1$, the result arrived at is

$$\frac{z}{Z} = -1 \qquad \text{or} \qquad z = -Z,$$

as was to be proved. (III. 23.)

The body of the Appendix contains, along with other matter which falls to be considered later, the statement and proof of propositions identical in essence but not in form with the following :—

$$(1) \qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & \overset{2}{\underset{1}{a}} & \dots & \overset{n-1}{\underset{1}{a}} & \overset{n}{\underset{1}{a}} \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & \dots & \overset{n-1}{\underset{2}{a}} & \overset{n}{\underset{2}{a}} \\ \dots & \dots & \dots & \dots & \dots \\ \overset{1}{\underset{n-1}{a}} & \overset{2}{\underset{n-1}{a}} & \dots & \overset{n-1}{\underset{n-1}{a}} & \overset{n}{\underset{n-1}{a}} \\ \overset{1}{T} & \overset{2}{T} & \dots & \overset{n-1}{T} & \overset{n}{T} \end{vmatrix} = 0, \qquad \text{(XLIX.)}$$

where

$$\overset{1}{T} = \underset{1}{m}\overset{1}{\underset{1}{a}} + \underset{2}{m}\overset{1}{\underset{2}{a}} + \dots + \underset{n-1}{m}\overset{1}{\underset{n-1}{a}}$$

$$\overset{2}{T} = \underset{1}{m}\overset{2}{\underset{1}{a}} + \underset{2}{m}\overset{2}{\underset{2}{a}} + \dots + \underset{n-1}{m}\overset{2}{\underset{n-1}{a}}$$

$$\dots\dots\dots\dots\dots$$

---

* More than a page is occupied in writing the expressions for z and Z.

$$(2)\qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & & & & \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & & & \\ \overset{1}{\underset{3}{a}} & \overset{2}{\underset{3}{a}} & \overset{3}{\underset{3}{a}} & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \\ \overset{1}{\underset{n}{a}} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \cdots\cdots & \overset{n}{\underset{n}{a}} \end{vmatrix} = \overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots\ldots \overset{n}{\underset{n}{a}}$$

The first of these is proved from first principles, and not by the immediate use of theorems XLVII., XLVIII. above. The second is proved by noting that any other term is got from the first

$$\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots\ldots \overset{n}{\underset{n}{a}}$$

by permutation of the under-indices, that any such permutation will introduce one or more elements whose upper-index exceeds the lower, and that such are all zero. (VI. 5.)

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, u.s.w. Von Ferd. Schweins. vi. + 666 pp. Heidelberg, 1825. Pp. 317–431; Theorie der Producte mit Versetzungen.]

With much of the preceding literature, Schweins, our next author, was thoroughly familiar. Cramer, Bézout, Hindenburg, Rothe, Laplace, Desnanot, and Wronski he refers to by name. The one notable investigator left out of his list is Cauchy, whose important memoir bearing date 1812 might have been known, one would think, to a writer who knew Desnanot's book of 1819 and Wronski's memoirs of 1810, 1811, &c. Still more curious is the omission of Vandermonde's name, whose memoir, as we have seen, is to be found in the very same volume as that of Laplace.

Schweins' portly volume consists of seven separate treatises. It is the third, headed *Theorie der Producte mit Versetzungen*, which deals expressly and exclusively with the subject of determinants. The treatise is logically arranged and carefully written. It opens with an introduction of 4 pp., the main part of which serves as a table of contents and as a guide to the theorems which the author considered his own. It consists of four Sections (Abtheilungen), subdivided into portions which we may call chapters, the first section containing five chapters, the second also five, the third one, and the fourth four.

Schweins' name for the functions is

Producte mit Versetzungen ; (xv. 6.)

his notation is a modification of Laplace's, viz., he uses

$$\Big\| \qquad \Big) \qquad\qquad \text{(vii. 6)}$$

where Laplace used simply

$$\Big(\qquad \Big);$$

and his definition is the same as Vandermonde's ; that is to say, he employs Bézout's law of recurring formation. His words at the outset are—

> "Die Bildungsweise der Producte, welche hier untersucht werden sollen, geben folgende Zeichen an :—
>
> $$\left\|\overset{a_1}{A_1}\right)=\overset{a_1}{A_1},$$
>
> $$\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\right)=\left\|\overset{a_1}{A_1}\right).\overset{a_2}{A_2}-\left\|\overset{a_2}{A_1}\right).\overset{a_1}{A_2},$$
>
> $$\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{A_3}\right)=\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\right).\overset{a_3}{A_3}-\left\|\overset{a_1}{A_1}\overset{a_3}{A_2}\right).\overset{a_2}{A_3}+\left\|\overset{a_2}{A_1}\overset{a_3}{A_2}\right).\overset{a_1}{A_3},$$
>
> $$\begin{aligned}\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{A_3}\overset{a_4}{A_4}\right)=&\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{A_3}\right).\overset{a_4}{A_4}-\left\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_4}{A_3}\right).\overset{a_3}{A_4}\\&+\left\|\overset{a_1}{A_1}\overset{a_3}{A_2}\overset{a_4}{A_3}\right).\overset{a_2}{A_4}-\left\|\overset{a_2}{A_1}\overset{a_3}{A_2}\overset{a_4}{A_3}\right).\overset{a_1}{A_4},\end{aligned}$$
>
> und allgemein
>
> $$\begin{aligned}\left\|\overset{a_1\,\cdots\cdots\,a_n}{A_1\,\cdots\,A_n}\right)=&(-)\left\|\overset{a_1\,\cdots\,a_{n-1}}{A_1\,\cdots\,A_{n-1}}\right).\overset{a_n}{A_n}+(-)^1\left\|\overset{a_1\cdots a_{n-2}\,a_n}{A_1\,\cdots\,A_{n-1}}\right)\overset{a_{n-1}}{A_n}+\cdots\\&+(-)^x\left\|\overset{a_1\cdots a_{n-x-1}\;a_{n-x+1}\cdots a_n}{A_1\,\cdots\cdots\cdots\,A_{n-1}}\right)\overset{a_{n-x}}{A_n}+\cdots+(-)^{n-1}\left\|\overset{a_2\,\cdots\,a_n}{A_1\cdots A_{n-1}}\right)\overset{a_1}{A_n}\end{aligned}$$
>
> oder
>
> $$\left\|\overset{a_1\,\cdots\cdots\,a_n}{A_1\,\cdots\,A_n}\right)=\Sigma(-)^x\left\|\overset{a_1\cdots a_{n-x-1}\;a_{n-x+1}\cdots a_n}{A_1\,\cdots\cdots\cdots\,A_{n-1}}\right).\overset{a_{n-x}}{A_n}\qquad x=0,1,\ldots,n-1.$$

The sequence of propositions as might be expected is not unlike that found in Vandermonde. The first six propositions are—

1. The under elements (A_1, A_2, &c.) being allowed to remain unchanged, the upper elements (a_1, a_2, . . .) are interchanged in every possible way to obtain the full development.

2. The sign preceding each term is dependent upon the number of interchanges of elements necessary to arrive at the term.

3. If two adjacent upper elements be interchanged, the sign of the *determinant* is altered.

4. If an upper element be moved a number of places to the right or left, the sign of the determinant is changed or not according as the number of places is odd or even.

5. If several upper elements change places, the sign of the determinant is altered or not according as the number is odd or even, which indicates how many cases there are of an element following one which in the original order it preceded.

6. If in any *term* the said number of pairs of elements in reversed order be even, the sign preceding the term must be positive; and if the number be odd, the sign must be negative.

The proof of the 3rd of these, which gave trouble to Vandermonde, is easily effected in what after all is Vandermonde's way, viz., by showing that the case for n elements follows with the help of the definition from the case for $n-1$ elements. (XI. 4.)

Schweins' 7th proposition is that there is an alternative recurring law of formation in which the under elements play the part of the upper elements in the original law, and *vice versa.* This amounts to saying in modern phraseology, that if a determinant has been shown to be developable in terms of the elements of a row and their complementary minors, it is also developable in terms of the elements of a column and their complementary minors. The proof is affected by the so-called method of induction, and is interesting both on its own account and from the fact that Cauchy's development in terms of binary products of a row and column turns up in the course of it. The character of the proof will be understood by the following illustrative example in the modern notation :—

By the original law of formation we have

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - a_2|b_1c_3d_4| + a_3|b_1c_2d_4| - a_4|b_1c_2d_3| ;$$

and, as the new law is supposed to have been proved for determinants of the 3rd order, it follows that

$$\begin{aligned}|a_1b_2c_3d_4| = a_1|b_2c_3d_4| &- a_2\{b_1|c_3d_4| - c_1|b_3d_4| + d_1|b_3c_4|\} \\ &+ a_3\{b_1|c_2d_4| - c_1|b_2d_4| + d_1|b_2c_4|\} \\ &- a_4\{b_1|c_2d_3| - c_1|b_2d_3| + d_1|b_2c_3|\} .\end{aligned}$$

Combining now by the original law the terms involving b_1 as a factor, the terms involving c_1, and those involving d_1, we obtain

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - b_1|a_2c_3d_4| + c_1|a_2b_3d_4| - d_1|a_2b_3c_4| ;$$

and thus prove that the new law holds for determinants of the 4th order. (VI. 6.)

Cauchy's development above referred to appears in the penultimate identity in the convenient form of one term $a_1|b_2c_3d_4|$ followed by a square array of 9 terms. The form in Schweins' is—

$$\left\| \begin{matrix} \mathrm{a}_1 \ldots\ldots \mathrm{a}_n \\ \mathrm{A}_1 \ldots \mathrm{A}_n \end{matrix} \right) = \left\| \begin{matrix} \mathrm{a}_1 \ldots\ldots \mathrm{a}_{n-1} \\ \mathrm{A}_1 \ldots \mathrm{A}_{n-1} \end{matrix} \right) \cdot \begin{matrix} \mathrm{a}_n \\ \mathrm{A}_n \end{matrix} + \Sigma_x \Sigma_y (-)^{x+y-1} \left\| \begin{matrix} \mathrm{a}_1 \ldots\ldots \mathrm{a}_{n-x-1} & \mathrm{a}_{n-x+1} \ldots\ldots \mathrm{a}_{n-1} \\ \mathrm{A}_1 \ldots \mathrm{A}_{n-y-1} & \mathrm{A}_{n-y+1} \ldots \mathrm{A}_{n-1} \end{matrix} \right) \cdot \begin{matrix} \mathrm{a}_{n-x} \\ \mathrm{A}_n \end{matrix} \cdot \begin{matrix} \mathrm{a}_n \\ \mathrm{A}_{n-y} \end{matrix}$$

Laplace's expansion-theorem is next taken up. To prepare the way a theorem in permutations is first given, the enunciation being as follows :—*If from* n *different elements every permutation of* q *elements be formed, and every permutation of* n – q *elements ; and if each of the latter be appended to all such of the former as have no elements in common with it, all the permutations of the whole* n *elements will be obtained.* Thus, if the permutations of 1 2 3 4 5, or say P (1 2 3 4 5), be wanted, we first take the permutations three at a time, viz.,

P(1 2 3), P(1 2 4), P(1 2 5), ... , P(3 4 5)

where 1 2 3, 1 2 4, 1 2 5, . . ., 3 4 5 are the orderly arranged combinations of three elements; secondly, we take the permutations two at a time, viz.,

P(1 2), P(1 3), P(1 4), ... , P(4 5) ;

and, thirdly, we append each of the two permutations included in P(4 5) to each of the six included in P(1 2 3), each of the two in

P(3 5) to each of the six in P(1 2 4), and so on. The identity here involved Schweins writes as follows, the only difference being that P is put instead of V (*Versetzungen*):—

$$\begin{aligned} P(1\,2\,3\,4\,5) = {} & P(1\,2\,3)\times P(4\,5) \\ & + P(1\,2\,4)\times P(3\,5) \\ & + P(1\,2\,5)\times P(3\,4) \\ & + P(1\,3\,4)\times P(2\,5) \\ & + P(1\,3\,5)\times P(2\,4) \\ & + P(1\,4\,5)\times P(2\,3) \\ & + P(2\,3\,4)\times P(1\,5) \\ & + P(2\,3\,5)\times P(1\,4) \\ & + P(2\,4\,5)\times P(1\,3) \\ & + P(3\,4\,5)\times P(1\,2). \end{aligned}$$

Another example is—

$$\begin{aligned} P(1\,2\,3\,4\,5\,6) = {} & P(1\,2\,3)\,.\,P(4\,5\,6) \\ & + P(1\,2\,4)\,.\,P(3\,5\,6) \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + P(3\,5\,6)\,.\,P(1\,2\,4) \\ & + P(4\,5\,6)\,.\,P(1\,2\,3). \end{aligned}$$

The proof consists in the assertion that no permutation can occur twice on the right-hand side, and in showing that the number of permutations which occur is the full number.

From this lemma Laplace's expansion-theorem is given as an immediate deduction. The passage (p. 335) is interesting, as the mode of enunciating the theorem approximates closely to that of modern writers, and has a certain advantage over Cauchy's, perfectly accurate, more general and more compact though the latter be.

> "Nach dieser Weise, alle Versetzungen zu bilden, welche wir hier zuerst bekannt machen, können auch die Summen der Producte mit Versetzungen und mit veränderlichen Zeichen in niedrigere Summen zerlegt werden, wenn bei jeder Versetzung nach der oben gefundenen Vorschrift das zugehörige Zeichen bestimmt wird; z. B.

$$
\begin{aligned}
\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix}\right)
&= \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&= \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_1 & a_2 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_3 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_5 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_2 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_3 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_4 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\left\|\begin{matrix} a_2 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_5 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_1 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_2 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_4 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_2 & a_3 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_4 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_3 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_5 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_2 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_4 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_2 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_3 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_4 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_3 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_2 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_3 & A_4 & A_5 \end{matrix}\right)\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix}\right).
\end{aligned}
$$

In der ersten Scheitelreihe sind die oberen und in der zweiten die unteren Elemente veränderlich; die Zeichen + und − befolgen das Gesetz in § 140. Eben so ist

(Another example is given.)

Wir wollen für diese Bildungsweise folgende allgemeine Zeichen wählen:

$$
\left\|\begin{matrix} a_1 \cdots\cdots a_n \\ A_1 \cdots A_n \end{matrix}\right) = \Sigma(-)^* \left\|\begin{matrix} (a_1\, a_2 \cdots\cdots a_n)^{(q)} \\ A_1 \cdots\cdots A_q \end{matrix}\right)\cdot\left\|\begin{matrix} (a_1\quad a_2 \quad \cdots\cdots a_n)^{(n-g)} \\ A_{q+1}\, A_{q+2} \cdots A_n \end{matrix}\right)
$$

und

$$
= \Sigma(-)^* \left\|\begin{matrix} a_1 \quad a_2 \cdots\cdots a_q \\ (A_1, A_2, \ldots, A_n)^{(q)} \end{matrix}\right.\cdot\left\|\begin{matrix} a_{q+1}\, a_{p+2} \cdots\cdots a_n \\ (A_1, A_2, \ldots, A_n)^{(n-q)} \end{matrix}\right)
$$

wo * nach dem Gesetze bestimmt werden muss, welches in § 140 gefunden ist." (XIV. 5.)

The one imperfection in this is in regard to the question of sign. It is implied that the sign to precede any product, say the product

$$\left|\overset{a_2}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3}\right)\cdot\left|\overset{a_1}{A_4}\,\overset{a_5}{A_5}\right)$$

is fixed by making it the same as the sign of the *term*

$$\overset{a_2}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3}\,\overset{a_1}{A_4}\,\overset{a_5}{A_5}\;;$$

but nothing is said as to how this ensures that the 11 other terms of the product shall have their proper sign.

Considerably less interest attaches to the next theorem dealt with,—Vandermonde's theorem regarding the effect of the equality of two upper or two lower elements. All that is fresh is the lengthy demonstration by the method of so-called induction. The identities immediately following from it by expansion Schweins expresses as follows:—

$$\Sigma(-)^x\left\|\begin{matrix} a_1 & \cdots\cdots\cdots & & a_{q-1}\,a_q\,a_{q+1} & \cdots & a_{n-1} \\ A_1 & \cdots & A_{x-1}\,A_{x+1} & \cdots\cdots & & A_n \end{matrix}\right)\cdot\overset{a_q}{A_x}=0$$

$$\Sigma(-)^x\left\|\begin{matrix} a_1 & \cdots\, a_{x-1}\,a_{x+1} & \cdots\cdots\cdots & a_n \\ A_1 & & A_{q-1}\,A_q\,A_{q+1}\,\cdots & A_{n-1} \end{matrix}\right)\cdot\overset{a_x}{A_q}=0$$

where $x=1\,,\,2\,,\,\ldots\,,\,n$. (XII. 10.)

This concludes the first chapter of the first section.

The second chapter deals with a most notable generalisation, and is worthy of being reproduced with little or no abridgment. The subject may be described as the transformation of an aggregate of products of pairs of determinants into another aggregate of similar kind. A special example of the transformation is taken to open the chapter with, the initial aggregate of products being in this case

$$|a_1b_2c_3d_4|\cdot|e_5f_6g_7| - |a_1b_2c_3e_4|\cdot|d_5f_6g_7|$$
$$+ |a_1b_2c_3f_4|\cdot|d_5e_6g_7| - |a_1b_2c_3g_4|\cdot|d_5e_6f_7|$$

Expanding the first factor of each product Schweins obtains

$$\left\{d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4|\right\}\cdot|e_5f_6g_7|$$
$$-\left\{e_4|a_1b_2c_3| - e_3|a_1b_2c_4| + e_2|a_1b_3c_4| - e_1|a_2b_3c_4|\right\}\cdot|d_5f_6g_7|$$
$$+\left\{f_4|a_1b_2c_3| - f_3|a_1b_2c_4| + f_2|a_1b_3c_4| - f_1|a_2b_3c_4|\right\}\cdot|d_5e_6g_7|$$
$$-\left\{g_4|a_1b_2c_3| - g_3|a_1b_2c_4| + g_2|a_1b_3c_4| - g_1|a_2b_3c_4|\right\}\cdot|d_5e_6f_7|\,.$$

He then combines the terms which contain $|a_1b_2c_3|$ as a factor, the terms which contain $|a_1b_2c_4|$ as a factor, and so forth, the result being by the law of formation,

$$|a_1b_2c_3|.|d_4e_5f_6g_7| - |a_1b_2c_4|.|d_3e_5f_6g_7| \\ + |a_1b_3c_4|.|d_2e_5f_6g_7| - |a_2b_3c_4|.|d_1e_5f_6g_7|.$$

The identity of this aggregate with the similar original aggregate constitutes the theorem.

The only point left in want of explanation in connection with it is the construction of the aggregate of products presented at the outset, it being, of course, impossible that any aggregate taken at will can be so transformable. A moment's examination suffices to show that when once the first product of all

$$|a_1b_2c_3d_4|.|e_5f_6g_7|$$

is chosen, the others are derivable from it in accordance with a simple law,—the requirements being (1) no change of suffixes, (2) the last letter of the first factor to be replaced by the letters of the second factor in succession, (3) the signs of the products to be + and − alternately. As for the first product of all, it is not difficult to see that the orders of the determinants composing it are quite immaterial. Instead of taking determinants of the 4th and 3rd orders, and producing by transformation an aggregate of products of determinants of the 3rd and 4th orders, we might have taken determinants of the $(n+1)$th and mth orders, applied the transformation, and obtained an aggregate of products of determinants of the nth and $(m+1)$th orders. This is the essence of Schweins' first generalisation. His own statement and proof of it leave little to be desired, and are worthy of examination in order that his firm grasp of the subject and his command of the notation may be known. He says (p. 345)—

"Die Reihe, welche in eine andere übertragen werden soll, sei

$$Q = \Sigma_x(-)^{x-1} \left\| \begin{matrix} a_1 & \ldots\ldots\ldots & & a_{n+1} \\ A_1 & \ldots\ldots & A_n & B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \ldots\ldots\ldots\ldots\ldots & b_m \\ B_1 & \ldots\, B_{x-1}\, B_{x+1} \ldots & B_{m+1} \end{matrix} \right)$$

$$\text{wo} \qquad x = 1, 2, \ldots\ldots, m+1 .$$

Der erste Factor wird nach 515 in niedere Summen aufgelöst

$$\left\| \begin{matrix} a_1 & \ldots\ldots\ldots & & a_{n+1} \\ A_1 & \ldots\ldots & A_n & B_x \end{matrix} \right) = \Sigma_y(-)^{n-y+1} \left\| \begin{matrix} a_1 & \ldots\, a_{y-1}\, a_{y+1} \ldots & a_{n+1} \\ A_1 & \ldots\ldots\ldots\ldots & A_n \end{matrix} \right) \cdot \overset{a_y}{B_x}$$

$$\text{wo} \qquad y = 1, 2, \ldots, n+1$$

wodurch die vorgegebene Reihe in folgende übergeht:

$$Q = \Sigma_x \Sigma_y (-)^{n+x-y} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \cdot \quad a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right) \cdot \overset{a_y}{B_x} \cdot$$

Es ist aber nach 522

$$\Sigma_x (-)^{m+1-x} \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right) \cdot \overset{a_y}{B_x} = \left\| \begin{matrix} b_1 \ldots\ldots b_m\, a_y \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right)$$

$$= (-)^m \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right);$$

folglich

$$Q = \Sigma_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) . \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right),$$

oder es ist

$$\Sigma_x (-)^{x-1} \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right)$$

$$= \Sigma_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right),$$

wo $x = 1, 2, \ldots\ldots, m+1,$

$y = 1, 2, \ldots\ldots, n+1;$

oder es ist

$$\left. \begin{array}{l} \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_2 \ldots\ldots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_2 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_3 \ldots\ldots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_3 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_2\, B_4 \ldots B_{m+1} \end{matrix} \right) \\ - \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ + (-)^m \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_{m+1} \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1 \ldots\ldots B_m \end{matrix} \right), \end{array} \right\}$$

$$= \left. \begin{array}{l} \left\| \begin{matrix} a_1 \ldots\ldots a_n \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n+1}\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \ldots a_{n-1}\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_n\, b_1 \ldots\ldots\ldots b_m \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \ldots a_{n-2}\, a_n\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n-1} b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right) \\ - \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ (-)^n \left\| \begin{matrix} a_2 \ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) . \left\| \begin{matrix} a_1\, b_1 \ldots\ldots\ldots b_m \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right) \end{array} \right\} . \qquad (\text{L.})$$

The further generalisation of which this is possible, and which Schweins effects, depends on the fact, that the law of formation twice used in proving the identity, is but the simplest case of Laplace's expansion-theorem, and that the said theorem can be similarly used in all its generality. In other words, instead of taking only *one* of the B's at a time to go along with the A's to form the first factors of the left-hand aggregate, we may take any fixed number of them. For example, out of six B's we may take every set of *three* to go along with two A's, and we shall have the aggregate

$$\begin{aligned}
&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\Big)\cdot\Big\|\overset{b_1}{B_4}\overset{b_2}{B_5}\overset{b_3}{B_6}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_4}\Big)\cdot\Big\|\overset{b_1}{B_3}\overset{b_2}{B_5}\overset{b_3}{B_6}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_6}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_5}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_4}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_5}\overset{b_3}{B_6}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_6}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_5}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_6}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_5}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_5}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_4}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_4}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_5}\overset{b_3}{B_6}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_6}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_5}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_6}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_5}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_5}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_4}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_5}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_6}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_5}\Big)\\
+&\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_5}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_4}\Big)-\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_4}\overset{a_4}{B_5}\overset{a_5}{B_6}\Big)\cdot\Big\|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_3}\Big),
\end{aligned}$$

—the sign of any term being determined by the number of inversions of order among the suffixes of all the B's of the term. In this particular case the first use of Laplace's expansion-theorem is to transform

$$\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\Big)$$

and the other similar determinants each into an aggregate of ten products, the two factors of any product in the expansion of

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_3 \end{matrix} \right)$$

being, as we should nowadays say, a minor formed from the first two rows and the complementary minor. In this way would arise 20 rows of 10 terms each, and these being combined by a second use of Laplace's expansion-theorem in columns of 20 terms each, the outcome would be an aggregate of 10 products, viz., the aggregate

$$\begin{aligned}
&\left\| \begin{matrix} a_1 & a_2 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_3 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
+&\left\| \begin{matrix} a_1 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
+&\left\| \begin{matrix} a_2 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_2 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
+&\left\| \begin{matrix} a_2 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_3 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
+&\left\| \begin{matrix} a_3 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right).
\end{aligned}$$

The following is Schweins' statement of this most general theorem:— (L. 2)

$$\begin{aligned}
&\Sigma(-)^{*} \left\| \begin{matrix} a_1 & \ldots\ldots\ldots & & \ldots\ldots & a_n \\ A_1 & \ldots & A_{n-q} \, B'_1 & \ldots & B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \ldots\ldots & b_{m-q} \\ B'_{q+1} & \ldots\ldots & B'_m \end{matrix} \right) \\
= &\Sigma(-)^{*} \left\| \begin{matrix} a'_1 & \ldots\ldots\ldots & a'_{n-q} \\ A_1 & \ldots\ldots\ldots & A_{n-q} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{n-q+1} \ldots a'_n \; b_1 \ldots b_{m-q} \\ B_1 \ldots\ldots\ldots B_m \end{matrix} \right).
\end{aligned}$$

The only points about it requiring explanation are the exact effect to be given to the symbol Σ, and the meaning of the dashes affixed to certain of the letters. The two symbols are connected with each other, the dashes not being permanently attached to the letters, but merely put in to assist in explaining the duty of the Σ. On the left-hand member of the identity, the two symbols indicate that the first term is got by dropping the dashes, and that from this first term another term is got, if we substitute for $B_1 \ldots . B_q$, some other set of q B's chosen from $B_1 \ldots . B_m$, and take the remaining B's to form the B's of the second determinant,—the two sets of

B's being in both cases first arranged in ascending order of their suffixes. On the other side of the identity, the use of the symbols is exactly similar, $n-q$ of the n upper elements $a_1, \ldots, a_n$ being taken for the first determinant of any term of the series, and the remainder for the second determinant. The number of terms in the series on the one side is evidently $m!/q!(m-q)!$ and on the other $n!/q!(n-q)!$

In the demonstration of the theorem greater fulness is evidently necessary than in the case of the previous theorem, the rule of signs in particular requiring attention. This Schweins' does not give. He merely tells the character of the first transformation, symbolising the expansion obtainable, and then says that a recombination is possible, giving the result.

The succeeding five pages (pp. 350–355) are devoted to evolving and stating special cases. This is by no means unnecessary work, as in the case of a theorem of so great generality it is often a matter of some trouble to ascertain whether a particular given result be really included in it or not. To students of the history of the subject the special cases are doubly interesting, because it is in them we may expect to find links of connection with the work of previous investigators.

The first descent from generality is made by putting some of the B's equal to A's, the theorem then being (L. 3)

$$\Sigma(-)^* \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots a_{p+s+q} \\ A_1 \ldots A_{p+s} B'_1 \ldots B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_{k+p} \\ B'_{q+1} \ldots B_{q+k} A_1 \ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} a'_1 \ldots\ldots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} \ldots a'_{p+s+q}\, b_1 \ldots\ldots b_{k+p} \\ B_1 \ldots\ldots\ldots\ldots B_{q+k} A_1 \ldots A_p \end{matrix} \right) .$$

If in addition to this specialisation, some of the b's be put equal to the a's, the result is (L. 4)

$$\Sigma(-)^* \left\| \begin{matrix} b_1 \ldots b_h\, a'_1 \ldots a'_{p+s-h} \\ A_1 \ldots\ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s-h+1} \ldots\ldots a'_{p+s-h+q},\ b_1 \ldots b_{h+k} \\ B_1 \ldots B_{h+k-p+q} A_1 \ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} b_1 \ldots b_h\, a_1 \ldots\ldots a_{p+s+q-h} \\ A_1 \ldots A_{p+s} B'_1 \ldots B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_{h+k} \\ B'_{q+1} \ldots B'_{q+h+k-p} A_1 \ldots A_p \end{matrix} \right)$$

—a notable theorem, which it would not be inappropriate to consider rather as a generalisation than as a special case of the theorem from which it is derived. Returning, however, to the preceding case, and putting $k=0$, we obtain (L. 5)

$$\left\|\begin{matrix}\mathrm{a}_1 \cdots\cdots\cdots\cdots\cdots \mathrm{a}_{p+s+q} \\ \mathrm{A}_1 \ldots \mathrm{A}_{p+s}\,\mathrm{B}_1 \ldots \mathrm{B}_q\end{matrix}\right) \cdot \left\|\begin{matrix}b_1 \cdots\cdots\cdots b_p \\ \mathrm{A}_1 \ldots\ldots \mathrm{A}_p\end{matrix}\right)$$

$$= \Sigma(-)^* \left\|\begin{matrix}\mathrm{a}'_1 \cdots\cdots \mathrm{a}'_{p+s} \\ \mathrm{A}_1 \ldots\ldots \mathrm{A}_{p+s}\end{matrix}\right) \cdot \left\|\begin{matrix}\mathrm{a}'_{p+s+1} \cdots\cdots\cdots \mathrm{a}'_{p+s+q}\, b_1 \cdots\cdots b_p \\ \mathrm{B}_1 \ldots\ldots \mathrm{B}_q\,\mathrm{A}_1 \ldots\ldots\ldots\ldots \mathrm{A}_p\end{matrix}\right) .$$

This may be viewed as an extension of Laplace's expansion-theorem to which it degenerates when p is put equal to 0. Though a comparatively very special identity it is considerably beyond anything attained by Schweins' predecessors. In fact, we only come upon something like known ground, when in descending further, we put in it $q=1$. The result thus obtained is

$$\left\|\begin{matrix}\mathrm{a}_1 \cdots\cdots\cdots\cdots \mathrm{a}_{p+s+1} \\ \mathrm{A}_1 \ldots\ldots \mathrm{A}_{p+s}\,\mathrm{B}_1\end{matrix}\right) \cdot \left\|\begin{matrix}b_1 \cdots\cdots\cdots\cdots b_p \\ \mathrm{A}_1 \ldots\ldots\ldots \mathrm{A}_p\end{matrix}\right)$$

$$= \Sigma(-)^* \left\|\begin{matrix}\mathrm{a}'_1 \cdots\cdots\cdots\cdots \mathrm{a}'_{p+s} \\ \mathrm{A}_1 \ldots\ldots\ldots \mathrm{A}_{p+s}\end{matrix}\right) \cdot \left\|\begin{matrix}\mathrm{a}'_{p+s+1}\, b_1 \cdots\cdots b_p \\ \mathrm{B}_1 \quad \mathrm{A}_1 \ldots\ldots \mathrm{A}_p\end{matrix}\right), \quad (\textsc{xlvi}.\ 2).$$

which closely resembles a theorem of Desnanot's. The difference between them consists in the fact that here the second factor on the left-hand side is *any* determinant of a lower order than the cofactor, whereas in Desnanot the second factor is a *minor* of the cofactor. A further specialisation, viz. putting $\mathrm{B}_1 = \mathrm{A}_{p+1}$, brings us to the result

$$\left.\begin{aligned} &\Sigma(-)^* \left\|\begin{matrix}\mathrm{a}'_1 \cdots\cdots \mathrm{a}'_{p+s} \\ \mathrm{A}_1 \ldots\ldots \mathrm{A}_{p+s}\end{matrix}\right) \cdot \left|\begin{matrix}\mathrm{a}'_{p+s+1},\, b_1 \cdots\cdots b_p \\ \mathrm{A}_1 \ldots\ldots\ldots \mathrm{A}_{p+1}\end{matrix}\right) = 0 , \\ &\text{or} \\ &\Sigma(-)^* \left\|\begin{matrix}b_1 \cdots\cdots\cdots b_{p+1} \\ \mathrm{A}_1 \ldots\ldots \mathrm{A}_p\,\mathrm{B}_1\end{matrix}\right) \cdot \left\|\begin{matrix}b_1 \quad b_2 \cdots\cdots b_{p+2} \\ \mathrm{B}'_2,\, \mathrm{B}'_3 \ldots\ldots \mathrm{B}'_{p+3}\end{matrix}\right) = 0 . \end{aligned}\right\} \quad (\textsc{xxiii}.\ 7).$$

The form here is that of a vanishing aggregate of products of pairs of determinants, and identities of this form we have already had to consider in dealing with Bézout, Monge, Cauchy, and Desnanot. To the last of these only does Schweins refer. His words are (p. 352)—

> "Wird in dieser Gleichung $s=2$ gesetzt, so entsteht folgende:—
>
> $$\Sigma(-)^* \left\|\begin{matrix}b_1 \cdots\cdots\cdots b_{p+1} \\ \mathrm{A}_1 \ldots \mathrm{A}_p\,\mathrm{B}'_1\end{matrix}\right) \cdot \left\|\begin{matrix}b_1 \quad b_2 \cdots\cdots b_{p+2} \\ \mathrm{B}'_2\,\mathrm{B}'_3 \ldots\ldots \mathrm{B}'_{p+3}\end{matrix}\right) = 0 ,$$
>
> wovon Desnanot einige ganz specielle Fälle gefunden hat, oder vielmehr der ganze Inhalt seiner Untersuchung ist in folgenden dreien Gleichungen begriffen

$$\Sigma(-)^{*}\left\|\begin{matrix} b_1 & b_2 & b_3 \\ A_1 & A_2 & B'_1 \end{matrix}\right) \cdot \left\|\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix}\right) = 0\,,$$

$$\Sigma(-)^{*}\left\|\begin{matrix} b_1 & b_2 \\ A_1 & B'_1 \end{matrix}\right) \cdot \left\|\begin{matrix} b_1 & b_2 & b_3 \\ B'_2 & B'_3 & B'_4 \end{matrix}\right) = 0\,,$$

$$\Sigma(-)^{*}\ \overset{b_1}{B'_1} \cdot \left\|\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix}\right) = 0\,,$$

welche mit ermüdender Weitläufigkeit bewiesen sind . . . . . ."

This statement is unfortunately not by any means accurate. As for the "ermüdende Weitläufigkeit," there can be no doubt about it, and to assert its existence is fair criticism; but to say that the whole of Desnanot's results are to be found in the three identities specified is a misrepresentation of the actual facts, and therefore quite unfair. The reader has only to turn back for a moment to our account of Desnanot's work, to verify the fact that the two most important general results attained by the latter (XXIII. 6 and XLVI.) are ignored by Schweins altogether.

The remaining paragraphs of the chapter are taken up with the very elementary case in which the products are three in number, and the theorem itself nothing more than one of the extensionals so lengthily dwelt upon by Desnanot, viz., the extensional

$$a_1|b_1c_2| \;-\; b_1|a_1c_2| \;+\; c_1|a_1b_2| \;=\; 0\,.$$

It is written in several forms, *e.g.*—

$$\begin{aligned}
&\left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots & a_{n+m} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;. & A_{n+m} \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots & a_{n+m+1} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m+1}B & \end{matrix}\right) \\
-&\left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots & a_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots & A_{n+m+1} \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots & a_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m} & B \end{matrix}\right) \\
+&\left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots & a_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m}B & \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots & a_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. & A_{n+m+1} \end{matrix}\right) = 0\,.
\end{aligned}$$

The next chapter, the third, concerns the solution of a set of linear equations, although according to the title its subject is the transformation of determinants into other determinants when the elements are connected by linear equations. It presents no new feature.

The fourth chapter deals with a special form of determinant, the consideration of which must therefore be deferred. Suffice it for the present to say, as an evidence of Schweins' grasp of the subject, that the solution of the problem attempted is complete and the result very interesting.

The fifth gives the solution of a problem on which the general Theory of Series is said to depend, the problem being the transformation of

$$\frac{\left\| \begin{matrix} \mathfrak{a}_1 & \mathfrak{a}_2 & \ldots\ldots\ldots\ldots & \mathfrak{a}_\infty \\ \mathrm{B} & \mathrm{A} & \ldots \mathrm{A}_{n-1}\mathrm{A}_{n+1} \ldots & \mathrm{A}_\infty \end{matrix} \right)}{\left\| \begin{matrix} \mathfrak{a}_1 & \mathfrak{a}_2 & \ldots\ldots\ldots\ldots & \mathfrak{a}_\infty \\ \mathrm{A}_1 & \mathrm{A}_2 & \ldots\ldots\ldots\ldots & \mathrm{A}_\infty \end{matrix} \right)}$$

into an unending series. The numerator, it will be observed, is of the order ∞ : the denominator is of the same order: and all the rows of the former except one occur in the latter. Indeed, if the first row of the numerator were deleted, and the n^{th} row of the denominator, there would be nothing to distinguish the one from the other. The subject is best illustrated by a special example in more modern notation. Recurring to the extensional above referred to as the concluding theorem of the second chapter, and taking the case where the factors are of the 4th and 5th orders, we manifestly have

$$|a_1b_2c_3e_4|.|a_1b_2c_3d_4f_5| - |a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5| + |a_1b_2c_3f_4|.|a_1b_2c_3e_4d_5| = 0,$$

from which, on dividing by $|a_1b_2c_3e_4|.|a_1b_2c_3e_4d_5|$, we obtain

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3e_4d_5|} - \frac{|a_1b_2c_3d_4|}{|a_1b_2c_3e_4|}.\frac{|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4d_5|} + \frac{|a_1b_2c_3f_4|}{|a_1b_2c_3e_4|} = 0.$$

Similarly

$$\frac{|a_1b_2c_3f_4|}{|a_1b_2e_3c_4|} - \frac{|a_1b_2c_3|}{|a_1b_2d_3|}.\frac{|a_1b_2e_3f_4|}{|a_1b_2e_3c_4|} + \frac{|a_1b_2f_3|}{|a_1b_2e_3|} = 0,$$

$$\frac{|a_1b_2f_3|}{|a_1e_2b_3|} - \frac{|a_1b_2|}{|a_1e_2|}.\frac{|a_1e_2f_3|}{|a_1e_2b_3|} + \frac{|a_1f_2|}{|a_1e_2|} = 0,$$

and $$\frac{|a_1f_2|}{|e_1a_2|} - \frac{a_1}{e_2}.\frac{|e_1f_2|}{|e_1a_2|} + \frac{f_1}{d_1} = 0,$$

the last fraction of each identity, be it observed, being the same as the first of the next with its sign changed. From the four by addition we have

$$\begin{aligned}\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3d_4e_5|} &= \frac{|a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4||a_1b_2c_3d_4e_5|}\\ &+ \frac{|a_1b_2c_3||a_1b_2e_3f_4|}{|a_1b_2e_3||a_1b_2c_3e_4|}\\ &+ \frac{|a_1b_2|.|a_1e_2f_3|}{|a_1e_2||a_1b_2e_3|}\\ &+ \frac{a_1}{e_1}.\frac{|e_1f_2|}{|a_1e_2|}\\ &+ \frac{f_1}{e_1}.\end{aligned}$$

The general result, as stated by Schweins, is that

$$(-)^{n+1}\frac{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m+1}\\ \mathrm{B}\ \mathrm{A}_1 \ldots \mathrm{A}_{n-1}\mathrm{A}_{n+1} \ldots \mathrm{A}_{n+m+1}\end{matrix}\right)}{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m+1}\\ \mathrm{A}_1 \ldots \ldots \ldots \ldots \ldots \mathrm{A}_{n+m+1}\end{matrix}\right)}$$

$$= \mathrm{L}^{(n)}_0.\mathrm{V}^{(n)} - \mathrm{L}^{(n)}_1.\mathrm{V}^{(n+1)} + \mathrm{L}^{(n)}_2.\mathrm{V}^{(n+2)} - \ldots.(-)^{m+1}\mathrm{L}^{(n)}_{m+1}.\mathrm{V}^{(n+m+1)},$$

where $$\mathrm{L}^{(n)}_m = \frac{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m-1}\\ \mathrm{A}_1 \ldots \mathrm{A}_{n-1}\mathrm{A}_{n+} \ldots \mathrm{A}_{n+m}\end{matrix}\right)}{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m-1}\\ \mathrm{A}_1 \ldots \ldots \ldots \ldots \ldots \mathrm{A}_{n+m-1}\end{matrix}\right)},$$

and $$\mathrm{V}^{(n+m)} = \frac{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m}\\ \mathrm{A}_1 \ldots \ldots \ldots \mathrm{A}_{n+m-1}\ \mathrm{B}\end{matrix}\right)}{\left\|\begin{matrix}\mathfrak{a}_1 \ldots \ldots \ldots \ldots \ldots \mathfrak{a}_{n+m}\\ \mathrm{A}_1 \ldots \ldots \ldots \ldots \ldots \mathrm{A}_{n+m}\end{matrix}\right)}. \quad \text{(LI.)}$$

Since the expression thus expanded is itself one of the L's, viz., $\mathrm{L}^{(n)}_{m+2}$—as is readily seen by transferring the B from the beginning to the end, and denoting it by A_{n+m+2},—and since $\mathrm{L}^{(n)}_0 = 1$, the identity may equally appropriately be written with $\mathrm{L}^{(n)}_{m+2}$ at the end of the right-hand member, and looked upon as the recurring law of formation of the L's in terms of the V's. This Schweins does, giving indeed the result of solving for $\mathrm{L}^{(n)}_1$, $\mathrm{L}^{(n)}_2$, . . .

The Second Section, consisting of five chapters, and extending to 30 pp., is devoted to a special form of determinants, viz., those already partly investigated by Cauchy, and afterwards known as alternants.

The Third Section, extending only to 4 pp., deals with another special form, whose elements are finite differences of a set of functions.

The Fourth Section, consisting of four chapters, and extending to 27 pp., has for its subject a third special form, foreshadowed by Wronski, the characteristic of which is that one of the indices denotes repetition of an operation involving differentiation.

When these Sections come to be considered in their proper places, it will be seen that very great credit is due to Schweins for his labours, and that he has been most undeservedly neglected. The fact that he had ever written on determinants was only brought to light in 1884:* and, so far as can be gathered, his treatise had no influence whatever either on the work of succeeding investigators, or in diffusing a knowledge of the subject.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233.]

[De singulari quadam duplicis Integralis transformatione. *Crelle's Journal*, ii. pp. 234–242.]

[Ueber die Pfaffsche Methode, eine gewöhnliche lineäre Differentialgleichung zwischen 2*n* Variabeln durch ein System von *n* Gleichungen zu integriren. *Crelle's Journal*, ii. pp. 347–357.]

We come here simultaneously on the names of a great mathematician and a great mathematical journal. Crelle's *Journal für die reine und angewandte Mathematik*, which began to appear at the end of the year 1825, and which without any of the symptoms of old age still survives, has rendered on more than one occasion important service towards the advancement of the theory of determinants. Its first contributor on the subject and one of its greatest was Jacobi. At a later date he published in the *Journal* an excellent monograph on Determinants; but even his earliest papers show that he had begun to find it a useful weapon of research.

In the first of the memoirs above noted, dealing with the subject of orthogonal substitution, constant use is, of course, made of the functions; but there is no special notation employed, nor indeed anything to indicate that the expressions used were members of a class having properties peculiar to themselves.

---

* v. *Phil. Mag.* for November: *An overlooked Discoverer in the Theory of Determinants.*

In the second memoir, which likewise is taken up with a transformation, but in which the sets of equations involve *four* unknowns, any special notation is still avoided. Expressions, readily seen to be determinants of the third order, are even not set down, because, as the author expressly states, they would be too lengthy. The last clause of the passage in which this statement occurs is noteworthy. The words are (p. 236)—

> "Dato systemate æquationum
>
> $$\begin{aligned} \alpha\ u + \beta\ x + \gamma\ y + \delta\ z &= m, \\ \alpha'\ u + \beta'\ x + \gamma'\ y + \delta'\ z &= m', \\ \alpha''u + \beta''x + \gamma''y + \delta''z &= m'', \\ \alpha'''u + \beta'''x + \gamma'''y + \delta'''z &= m''', \end{aligned}$$
>
> "ponamus earum resolutione erui:
>
> $$\begin{aligned} \mathrm{A}m + \mathrm{A}'m' + \mathrm{A}''m'' + \mathrm{A}'''m''' &= u, \\ \mathrm{B}m + \mathrm{B}'m' + \mathrm{B}''m'' + \mathrm{B}'''m''' &= x, \\ \mathrm{C}m + \mathrm{C}'m' + \mathrm{C}''m'' + \mathrm{C}'''m''' &= y, \\ \mathrm{D}m + \mathrm{D}'m' + \mathrm{D}''m'' + \mathrm{D}'''m''' &= z. \end{aligned}$$
>
> "Valores sedecim quantitatum A, B, etc., supprimimus eorum prolixitatis causa; in libris algebraicis passim traduntur, et algorithmus, cuius ope formantur, hodie abunde notus est."

On the next page, in eliminating D, D′, D″, D‴ from the set of equations

$$\begin{aligned} 0 &= \mathrm{D}(a-x) + \mathrm{D}'b' + \mathrm{D}''b'' + \mathrm{D}'''b''', \\ 0 &= \mathrm{D}b' + \mathrm{D}'(a'+x) + \mathrm{D}''c''' + \mathrm{D}'''c'', \\ 0 &= \mathrm{D}b'' + \mathrm{D}'c''' + \mathrm{D}''(a''+x) + \mathrm{D}'''c', \\ 0 &= \mathrm{D}b''' + \mathrm{D}'c'' + \mathrm{D}''c' + \mathrm{D}'''(a'''+x), \end{aligned}$$

he arranges the resultant as one would now do who had expanded it from the determinant form according to products of the elements of the principal diagonal, viz., he says (p. 238)—

"Fit illa, eliminationis negotio rite instituto

$$\begin{aligned}0 = {} & (a-x)(a'+x)(a''+x)(a'''+x) \\ & - (a-x)(a'+x)c'c' - (a-x)(a''+x)c''c'' - (a-x)(a'''+x)c'''c''' \\ & - (a''+x)(a'''+x)b'b' - (a'''+x)(a'+x)b''b'' - (a'+x)(a''+x)b'''b''' \\ & + 2c'c''c'''(a-x) + 2c'b''b'''(a'+x) \qquad \text{(LII.)} \\ & + 2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x) \\ & + b'b'c'c' + b''b''c''c'' + b'''b'''c'''c''' - 2b'b''c'c'' - 2b''b'''c''c''' - 2b'''b'c'''c'.\end{aligned}$$

From the next paragraph we learn his sources of information, and infer that as yet Cauchy's memoir was unknown to him. The first sentence is (p. 239)—

> "Inter sedecim quantitates α, β, etc. et sedecim, quæ ex iis derivantur, A, A′, etc. plurimæ intercedunt relationes perelegantes, quæ cum analystis ex iis, quæ Laplace, Vandermonde, in commentariis academiæ Parisiensis A. 1772 p. ii., Gauss in disquis. arithm. sectio V., J. Binet in vol. ix. diariorum instituti polytechnici Pariensis, aliique tradiderunt, satis notæ sint, paucas tantum referam, quæ casu nostro speciali ope æquationum (IV) facile ex iis derivantur."

The third memoir is by far the most important to us. In the course of the investigation a new special form of determinants, afterwards so well known by the designation *skew* determinants, turns up; and three pages are devoted to an examination of the final expanded form of it. This examination, we cannot, of course, now enter upon; but it is of importance to note that in it Jacobi takes the step of adopting the name *determinant*,—a fact which doubtless was decisive of the fate of the word. The adoption thus made (although stated to be from Gauss), and the occurrence of the words "Horizontalreihen," "Verticalreihen," make it probable that Cauchy's memoir had now come to his notice.

Table—*Showing the Advance of the Theory of Determinants from* 1693 *to* 1812.

			Leibnitz, 1693.	Cramer, 1750.	Bézout, 1764.	Vandermonde, 1771.	Laplace, 1772.	Lagrange, 1773.	Bézout, 1779.	Hindenburg, 1784.	Rothe, 1800.	Gauss, 1801.	Monge, 1809.	Hirsch.	Binet, 1811.	Prasse.	Wronski, 1812.	Binet.	Cauchy.
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